Convergence Analysis of Active Noise Control Systems Using Root Locus Theory

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***Abstract*—This paper applies root locus theory in order to conduct a new convergence analysis for the stochastic FxLMS algorithm, without any simplifying assumption regarding the secondary path. The main steps to sketch the root locus of the FxLMS adaptation process are developed. Using the obtained root locus plot, the upper bound for the adaptation step-size beyond which the adaptation process becomes unstable is derived. The proposed bound applies to moving average secondary systems, while previously proposed ones only apply to pure delay secondary systems. Results obtained from this study are found to agree very well with those obtained from the numerical analysis and simulation results.**

***Index Terms*—Active noise control, FxLMS algorithm, convergence analysis, root locus theory**

I. Introduction

Although *Filtered-x LMS* Algorithm (FxLMS) is widely used in different applications of adaptive filtering, its convergence analysis is still an active area of research. Among available FxLMS convergence analyses, only a few analyses have intended to derive a convergence condition for the adaptation process. Besides, available convergence conditions are only accurate for simplified cases with pure delay secondary paths or narrow-band noise signals. Long [1] derived a convergence condition for the FxLMS under assumptions that the secondary path is a pure delay system and the noise is a broad-band white signal. A similar condition has been independently reported by Elliott [2] based on simulation results. Bjarnason [3] conducted another FxLMS convergence analysis and derived a FxLMS convergence condition under assumptions that the secondary path is a pure delay system and the noise signal is a stochastic narrow-band or stochastic broad-band white signal. Both Long and Bjarnason's analyses were conducted under *Independence Assumption* [4] stating that consecutive vectors of the input signal are statistically independent. In 2006 Vicente [5] derived another FxLMS convergence condition assuming that the secondary path is a pure delay system and the noise signal is a narrow-band signal. Xiao [6] extended Vicente's analysis and derived a FxLMS convergence condition for the case that the secondary path is a *Moving Average* (MA) process and the noise signal is multi-tonal. However, as reported by Xiao, in his work simulation results were not in good agreement with theoretical results. Although the convergence analyses carried out in the above mentioned studies involve sophisticated mathematics but developing a FxLMS convergence condition for a MA secondary path and stochastic noise signal has not directly treated. This paper applies root locus theory in order to conduct a new FxLMS convergence analysis for a MA secondary path and stochastic white noise signal. The rest of paper is organized as follows. Section 2 gives mathematical description of FxLMS convergence behavior. Section 3 develops necessary steps to sketch FxLMS root locus plot. Section 4 conducts a convergence analysis using the root locus plot obtained in Section 3. Finally, Section 5 gives concluding remarks.

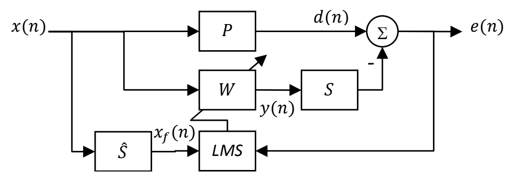

Figure 1. Block diagram of the FxLMS algorithm

II. System Model

As shown in figure 1 , the FxLMS minimizes the residual error $e(n)$ by adjusting the adaptive filter W followed by the secondary path S. The output signal $y(n)$ can be estimated as

$$y(n) = \mathbf{w}(n)^T \mathbf{x}(n), \tag{1}$$

where $\mathbf{x}(n)$ is a $L \times 1$ tap vector of the noise signal $x(n)$:

$$\mathbf{x}(n) = [x(n), x(n-1), ..., x(n-L+1)]^T, \tag{2}$$

and $\mathbf{w}(n)$ is adaptive weight vector defined as

$$\mathbf{w}(n) = [w_0(n), w_1(n), ..., w_{L-1}(n)]^T. \tag{3}$$

In order to minimize $e(n)$, the FxLMS updates $\mathbf{w}(n)$ using

$$\mathbf{w}(n+1) = \mathbf{w}(n) + \mu e(n) \mathbf{x}_f(n), \tag{4}$$

where μ is adaptation step-size and $\mathbf{x}_f(n)$ is obtained by filtering $\mathbf{x}(n)$ using an estimate of the secondary path, $\hat{S}$. Assuming that $\hat{S}$ is a MA process of length Q with parameters $s_0, s_1, ..., s_{Q-1}$, $\mathbf{x}_f(n)$ can be written as

$$\mathbf{x}_f(n) = \sum_{q=0}^{Q-1} s_q \mathbf{x}(n-q). \tag{5}$$

In [3], it is show that if $\mathbf{R}$ denotes *Auto-Correlation Matrix* (ACM) of $\mathbf{x}(n)$ and $\mathbf{p}$ denotes cross-correlation vector of $\mathbf{x}(n)$

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and $d(n)$, then the optimum solution for $\mathbf{w}(n)$ is

$$\mathbf{w}_{opt} = \mathbf{R}^{-1}\mathbf{p}. \tag{6}$$

The ACM matrix $\mathbf{R}$ can be decomposed to

$$\mathbf{R} = \mathbf{F}\boldsymbol{\Lambda}\mathbf{F}^{T}, \tag{7}$$

where $\mathbf{F}$ is modal Eigenvectors matrix and Λ is a diagonal matrix with eigenvalues $\lambda_0, \lambda_1, ..., \lambda_{L-1}$. Using $\mathbf{F}$, rotated input vector, $\mathbf{z}(n)$, is defined as

$$\mathbf{z}(n) \triangleq \mathbf{F}^{T}\mathbf{x}(n). \tag{8}$$

Also, rotated weight misalignment vector, $\mathbf{c}(n)$, is defined as:

$$\mathbf{c}(n) \triangleq \mathbf{F}^{T}\left[\mathbf{w}(n) - \mathbf{w}_{opt}\right], \tag{9}$$

In [3], the mean behavior of $\mathbf{c}(n)$ is described as

$$\bar{\mathbf{c}}(n+1) = \bar{\mathbf{c}}(n) - \mu\Lambda\sum_{q=0}^{Q-1} s_q^2\bar{\mathbf{c}}(n-q) \tag{10}$$

where $\bar{\mathbf{c}}(n)$ denotes statistical expectation of $\mathbf{c}(n)$.

III. Root Locus of The FxLMS Algorithm

Taking z-transform from (10) results in the following characteristic equation.

$$(z-1)\,1_{L\times 1} + \mu\Lambda\sum_{q=0}^{Q-1} s_q^2 z^{-q} = 0_{L\times 1}. \tag{11}$$

As matrix Λ is diagonal with $\lambda_0, \lambda_1, ...\lambda_{L-1}$ on its diagonal, (11) can be split into L independent equations that are

$$z - 1 + \mu\lambda_l\sum_{q=0}^{Q-1} s_q^2 z^{-q} = 0 \qquad l = 0, \cdots L-1. \tag{12}$$

These equations can be expressed as

$$1 + \mu\lambda_l\frac{G(z)}{z^{Q} - z^{Q-1}} = 0, \tag{13}$$

where $G(z)$ is

$$G(z) = s_0^2 z^{Q-1} + s_1^2 z^{Q-2} + ... + s_{Q-1}^2. \tag{14}$$

For the FxLMS stability, all the roots of (13) should be inside the unit circle. Since increasing the scalar parameter $\mu\lambda_l$ moves the roots out of the unit circle, it is sufficient to examine the roots only when $\lambda_l = \lambda_{max}$, where λ_{max} is the maximum value of $\{\lambda_0, \lambda_1, ...\lambda_{L-1}\}$. Therefore, the FxLMS stability involves the stability of

$$1 + \mu\lambda_{max}\frac{G(z)}{z^{Q} - z^{Q-1}} = 0 \tag{15}$$

Now, let us define the open loop transfer function $H(z)$ as

$$H(z) = \mu\lambda_{max}\frac{G(z)}{z^{Q} - z^{Q-1}} \tag{16}$$

and re-express (15) as

$$1 + \mu\lambda_{max}H(z) = 0 \tag{17}$$

According to the root locus concepts [7], points of the root locus should satisfy the magnitude criterion,

$$\mu = \frac{1}{\lambda_{max}\left|H(z)\right|}, \tag{18}$$

and the phase criterion,

$$\measuredangle H(z) = 180, \tag{19}$$

where $|.|$ and $\measuredangle.$ denote magnitude and angle. The root locus consists of a number of branches; for $\mu = 0$, the roots of (13) are located on the start points of these branches and the roots move on the branches as μ increases from zero to infinity. The number of branches is equal to the number of poles of $H(z)$. According to (16), $H(z)$ has Q poles. Thus the FxLMS root locus has Q branches. Let us name these branches $B_1, B_2, ..., B_Q$. Based on the root locus concepts [7], the following steps to sketching $B_1, B_2, ..., B_Q$ are proposed.

Step#1 (Direction of branches): Branches of the root locus start at poles of open loop system and approach the zeroes of this system. As the number of zeros is smaller than the number of poles branches for excess poles approach infinity. Generally, $H(z)$ has $Q-1$ zeros but once $s_0 = s_1 = ... = s_{Q_0-1} = 0$ the number of zeroes reduces to $Q - Q_0 - 1$. Also, $H(z)$ has a repeated pole of order $Q-1$ at the origin and a single pole at $z = 1$. Therefore, it can be assumed that B_1 starts at $z = 1$ and $B_2, B_3, ..., B_Q$ start at the origin. Depending on the location of the zeros of $G(z)$ and asymptotes of the root locus, a branch either ends at a zero or approaches an asymptote.

Step #2 (Departure angles from poles): From the angle criterion, it can be shown that the departure angle of B_1 from its start point at $z = 1$ is $\theta_1 = 180°$ and those of other branches from $z = 0$ are

$$\theta_q = \frac{2(q-2)}{Q-1}\pi \qquad q = 2, 3, ..., Q \tag{20}$$

where θ_q is the departure angle of the q-th branch from its start point.

Step #3 (Asymptotes): The FxLMS root locus has $Q_0 + 1$ asymptotes. These asymptotes originate on the real axis at the centroid point x_A given by:

$$x_A = \frac{\sum\left[poles\,of\;H(z)\right] - \sum\left[zeros\,of\;H(z)\right]}{Q_0 + 1} \tag{21}$$

and form angles with respect to the real axis of:

$$\alpha_k = \frac{(2k+1)}{Q_0+1}\pi \qquad k = 0, 1, ..., Q_0 \tag{22}$$

For $H(z)$ given by (16)

$$\sum\left[poles\,of\;H(z)\right] = 1 \tag{23}$$

$$\sum\left[zeros\,of\;H(z)\right] = \sum\left[zeros\,of\;G(z)\right] \tag{24}$$

From (14) we obtain

$$\sum\left[zeros\,of\;G(z)\right] = -\frac{s_{Q_0+1}^2}{s_{Q_0}^2} \tag{25}$$

Substituting (23)-(25) into (21), x_A is computed as:

$$x_A = \frac{1 + \frac{s^2_{Q_0+1}}{s^2_{Q_0}}}{Q_0 + 1} \tag{26}$$

Step #4 (Real axis segments): The real segment of the root locus always lies in a section of the real axis to the left of an odd number of poles and zeros. Since all the coefficients of $G(z)$ are positive, $G(z)$ and accordingly $H(z)$ can only have complex conjugate zeros in the right side of the imaginary axis. Therefore, there are always an even number of zeros to the right side of the imaginary axis. On other hand, there is a single pole at $z = 1$; therefore the only positive real segment of the FxLMS root locus always lies in the interval $[0, 1]$. The FxLMS root locus may lies in some sections of the negative real axis, depending on the order of the repeated zero at the origin and the location of the negative real zeros of $G(z)$.

Step #5 (Break point): Breakaway points occur when branches of the root locus coincide. It can be found that B_1 and B_2 intersects each others at a break point on the real axis. In the following, the location of this point is approximately calculated. This point satisfies

$$\frac{\partial}{\partial z}\frac{1}{H(z)}\bigg|_{z=x_B} = 0 \tag{27}$$

Substituting (16) into this equation, results in

$$\frac{G'(x_B)}{G(x_B)}\left(x_B^2 - x_B\right) - Q\left(x_B - 1\right) = 0 \tag{28}$$

Assuming that the answer is close to $z = 1$, the following approximation can be made:

$$\frac{G'(x_B)}{G(x_B)} \approx \frac{G'(1)}{G(1)} \tag{29}$$

From (14), the following equation can be obtained:

$$z^{-Q+1}G(z) = \sum_{q=0}^{Q-1} s_q^2 z^{-q} \tag{30}$$

Setting $z = 1$ in this equation results in:

$$G(1) = \sigma_s^2 \tag{31}$$

Differentiating of both sides of (30) results in:

$$-(Q-1)\,z^{-Q}G(z) + z^{-Q+1}G'(z) = -\sum_{q=0}^{Q-1} q s_q^2 z^{-q-1}$$

Setting $z = 1$ in this equation results in:

$$-(Q-1)\,G(1) + G'(1) = -\tau_s^2 \tag{32}$$

where:

$$\tau_s^2 \equiv \sum_{q=0}^{Q-1} q s_q^2$$

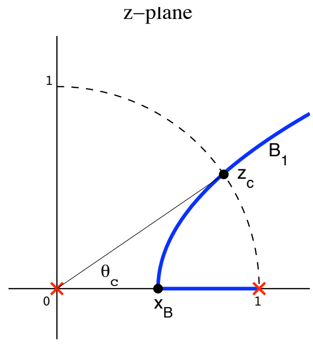

Figure 2. Typical behavior of the dominant pole of the FxLMS

Using (31) leads to rewrite the above equation as:

$$G'(1) = (Q-1)\,\sigma_s^2 - \tau_s^2 \tag{33}$$

Dividing (33) by (31) results in:

$$\frac{G'(1)}{G(1)} = Q - \Delta_{eq} - 1 \tag{34}$$

where Δ_{eq} is defined as follows:

$$\Delta_{eq} = \frac{\tau_s^2}{\sigma_s^2} \tag{35}$$

Substituting (34) into (28) results in:

$$x_B^2 + \frac{2Q - \Delta_{eq} - 1}{Q - \Delta_{eq} - 1}x_B + \frac{Q-1}{Q - \Delta_{eq} - 1} = 0 \tag{36}$$

This equation can be rewritten as follows:

$$(x_B - 1)^2 - \frac{\Delta_{eq} + 1}{Q - \Delta_{eq} - 1}x_B + \frac{\Delta_{eq}}{Q - \Delta_{eq} - 1} = 0 \tag{37}$$

Since $(x_B - 1)^2 \approx 0$, x_B can be approximately obtained as:

$$x_B \approx \frac{\Delta_{eq}}{\Delta_{eq} + 1} \tag{38}$$

IV. Stability Analysis

Branches of the FxLMS root locus has some typical properties. For example, it can be always seen that B_1 starts at $z = 1$ and moves on the real axis. Once B_1 reach the breakaway point x_B, it leaves the real axis. This branch may ends at a complex zero of $H(z)$ or approach the first asymptote of the root locus. This typical trajectory of B_1 is shown in Figure 2 . Also, it can be always seen that B_2 starts at $z = 0$ and moves on the positive real axis. Once B_2 reach the breakaway point x_B, it leaves the real axis in such a way that points of this branch are complex conjugates of those of B_1. This branch may end at a zero of $H(z)$, which is the complex conjugate of the end point of B_1, or approach the last asymptote of the locus. Other branches start at $z = 0$ and moves towards the unit circle in order to end at the zeros of $H(z)$ or approach the asymptotes. One example of the FxLMS root locus for a particular secondary path is given in Figure 3 .

Since x_B is close to z=1, it is expected that the unit circle is

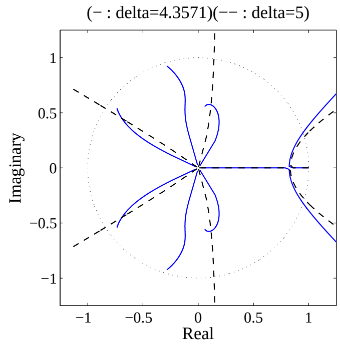

Figure 3. Root locus of the FxLMS algorithm with $S(z) = z^{-5}$ and $S(z) = z^{-3} + z^{-4} + z^{-5} + 0.5z^{-6} + 0.5z^{-7}$

closer to the root moving on B_1 than the other roots. Therefore, the dominant root of the system is located on B_1. Consequently, it is expected that the critical point is the intersection of B_1 and the unit circle. In Figure 2, the critical point is shown by z_c. Value of μ at the critical point can be considered as the upper bound of step-size beyond wich the system becomes unstable (μ_{max}). At the first look, calculation of μ_{max} is not analytically possible using information provided by the root locus.

It is found out that μ_{max} has a direct relationship with the distance of x_B to $z = 1$. In other words, the closer distance between x_B and $z = 1$ the smaller μ_{max}. Considering that $x_B = \Delta_{eq}/\Delta_{eq}+1$, this statement can be re-expressed as: the greater Δ_{eq}, the the smaller μ_{max}. In the following we explain this result using a particular example. For example, for the FxLMS algorithm with $S(z) = z^{-5}$, the value of μ_{max} is smaller than the case with $S_1(z) = z^{-3}+z^{-4}+z^{-5}+0.5z^{-6}+0.5z^{-7}$. This is because according to (35), for $S(z) = z^{-5}$ we have $\Delta_{eq} = 5$ and for $S(z) = S_1(z)$ we have $\Delta_{eq} = 4.3571$. Therefore, convergence condition of the FxLMS algorithm with $S(z) = z^{-5}$ can be considered as a sufficient condition for the convergence of the FxLMS algorithm with $S(z) = S_1(z)$. Figure 3 compares root loci of the FxLMS algorithm with these two secondary paths.

In general case, convergence condition of the FxLMS algorithm with $S(z) = z^{-\triangle}$ is a sufficient condition for the convergence of the FxLMS algorithm with a MA secondary path whose equivalent delay is Δ_{eq} if $\triangle$ is the least integer number greater than Δ_{eq} (or $\lceil\Delta_{eq}\rceil$). From [1], the convergence condition of the FxLMS algorithm with pure delay secondary path $z^{-\triangle}$ is

$$\mu < \frac{1}{(L + 2\Delta) P_{x_f}} \tag{39}$$

where P_{x_f} is power of the filtered input signal. Therefore, the sufficient condition for the convergence of the FxLMS with a general secondary path is

$$\mu < \frac{1}{(L + 2\lceil\Delta_{eq}\rceil) P_{x_f}} \tag{40}$$

where Δ_{eq} is the equivalen delay of the secondar path, given in Eq. 35. Note that the proposed convergence condition applies to a MA secondary path, but previosuly derived one could only apply to pure delay secondary paths.

V. Simulation Results

To verify the validity of the theoretical result, several computer simulation with different secondary paths have been carried out. As stated in [3] by Bjarnason, "the measurement of the stability bounds is a difficult matter and the results of the measurement have to be taken with caution. They can be regarded as guidelines for predicting the behavior of the algorithm for a white input." However, in the following, we show the validity of the proposed convergence condition by estimating convergence probability of the FxLMS algorithm. For this purpose, we selected 4 different secondary paths and calculated their μ_{max}. We simulated the FxLMS algorithm for each secondary path when μ is $0.85\mu_{max}, 0.9\mu_{max}, 0.95\mu_{max}, \mu_{max}, 1.05\mu_{max}$ and $1.1\mu_{max}$. For each case we repeated simulation for 50 times with different random white signals generated by the computer. Therefore, we had $4 \times 50 = 200$ simulation experiments for each μ(1200 experiments in total). For each μ, the number of stable experiments in percentage can be interpreted as the convergence probability of the FxLMS algorithm. When $\mu < 0.9\mu_{max}$ all experiments converge. It can be also found that when $\mu < \mu_{max}$ more than 95% of the experiments converge but when μ increases to $1.1\mu_{max}$ the convergence probability is approximately zero. Therefore the proposed condition is a reliable condition for the convergence of the FxLMS algorithm.

VI. Conclusion

Based on the root locus theory, a novel analytical convergence condition for the FxLMS algorithm is proposed and verified. Compared to the previously derived convergence conditions, the proposed condition applies to more general cases. It is shown that the location of the break point of the root locus plot restricts the upper bound of the convergence condition. Simulation results with 1200 different cases shows that this condition is a reliable condition and close to a sufficient condition for the convergence.

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