A New Non-Uniform Quantization Method Based on Distribution of Compressive Sensing Measurements and Coefficients Discarding

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Abstract—**Compressive sensing (CS) is a new method of sampling and compression which has great advantage over previous signal compression techniques. However, its compression ratio is relatively low compared with most of the current coding standards, which means a good quantization method is very important for CS. In this paper, a new method of non-uniform quantization is proposed based on the distribution of CS measurements and coefficients discarding. Firstly, the magnitude of CS measurements is estimated and the low probability measurements are discarded because of their high quantization error. It should be noted that the dropped measurements almost take no effect on the recovery quality because of the equal-weight property of CS samples. Then a non-linear quantize function based on the distribution of sensed samples is proposed, by which those remained measurements are quantized. The experimental results show that the proposed method can obviously improve the quality of reconstructed image compared with previous methods in terms of the same sampling rate and different reconstruction algorithms.**

I. INTRODUCTION

Compressed sensing (CS) is a new technology which has been studied widely in many fields for its good performance on sampling and compression. CS can compensate for losses of information through the erasure channel simply with oversampling [1]. However, due to the requirement of transmission, quantization has played a key role in the performance of compressed sensing. Recently, several methods of CS quantization have been studied. For example, Wei Dai, Hoa Vinh Pham et al. proposed scalar and vector quantization for CS and studied their properties respectively [2]. Ulugbek Kamilov et al. proposed an optimal method of reconstruction for quantized measurements [3]. An adaptive quantization scheme was proposed in [4], but this scheme will enhance the complexity of the front-end equipment greatly. A progressive quantization scheme was proposed in [5], which makes the complexity of recovery exponentially increase as a sacrifice. Previous quantization methods mainly focused on scalar and vector quantization. The distribution of CS measurements is seldom researched. This paper presents a method of non-uniform quantization for image transmission based on the distribution of CS measurements and coefficients discarding. In this method, a normalized non-uniform compression function is designed according to the distribution of CS measurements and some measurements which have large quantization errors are discarded to reduce the quantization range. Comparing with previous method, the reconstruction quality is enhanced obviously but the complexity of the quantization system increases little.

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II. DISTRIBUTION OF CS MEASUREMENTS

A. *Distribution of CS Measurements*

The distribution of CS measurements has a relationship with the measurement matrix and the sparse domain. In this paper we assume that the sparse transformation is DCT and the rows of measurement matrix are randomly selected from the unit matrix, meanwhile, the size of the image is 256×256. In practical application, images are often divided into small blocks so that they can be easily processed. In this paper, we set the block size to be 32×32.

Most of signals have certain natural characteristics both in time domain and frequency domain such as speech signal. After transformed to DCT domain, the coefficients of the image signal also has regular pattern. The statistical characteristic of DCT coefficients has already been studied. We can model the distribution of the coefficients with the Generalized Gaussian density Function (GGF). Fig. 1 is the kernel density of measurements of Lena. In the figure, the measurements are composed of two parts, the high frequency part and the low frequency part. Measurements which have small values belong to the high frequency part while those have larger values belong to the low frequency part. Based on the analysis above, we can quantize the two parts correspondingly. As can be seen from Fig.1, the two parts approximately obey the normal distribution. It is obvious that the distribution of CS measurements retained the property of original DCT coefficients distribution, which is Generalized Gaussian Distribution (GGD). Here the CS measurements are part of DCT coefficients which are randomly selected. No matter how much is the sampling rate, such extraction retained the distribution characteristics of the original information as long as the sampling matrix Φ satisfies the *Restricted Isometry Property* (RIP) [6]. So we can get all the information at the receiver to calculate the original value.

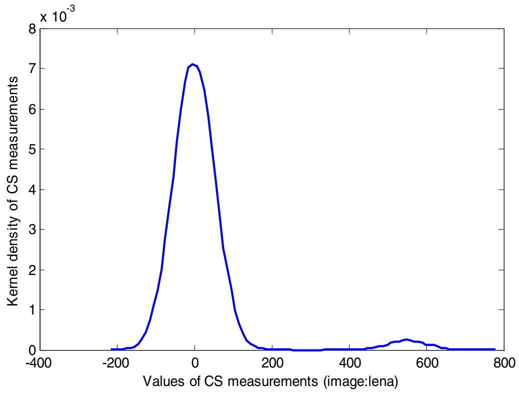

Fig. 1 Distribution of measurements of Lena.

B. *Theoretical Analysis*

Assume that the transformed values θ_i are independent and identically distributed (i.i.d.) random variables, and its probability density is $f(\theta)$ at the same time. So $y_i = \varphi_i \theta$ ($1 \le i \le M$) is a linear transformation of θ. Here φ_i is the i th row of sampling matrix $\Phi_{m \times n}$ which is randomly selected from a measurement matrix $\Phi_{n \times n}$, θ is the vector consisted of θ_i and its length is N, thus y_i is a measurement and it can be seen as a statistic of θ. In probability theory, if a probability distribution of random variables is determined, then the distribution of the statistic is also fixed, but it is very difficult to compute the probability density of y_i. We can estimate the average density of it. Assume it is $\overline{p(\mathrm{f})}$, because φ_i is a row random selected, we consider uniform probability. Expression of the mathematical expectation of probability density of y_i is derived as following,

$$\overline{p_j(\mathrm{f})} = \frac{1}{M}\sum_{i=1}^{M} p_i(\mathrm{f}) \tag{1}$$

Here $\overline{p_j(\mathrm{f})}$ is average density of one case which is selected from C_N^M cases, and $p_i(\mathrm{f})$ is probability density corresponding to the i th row of sampling matrix $\Phi_{m \times n}$. Thus,

$$\overline{p(\mathrm{f})} = \frac{1}{C_N^M}\sum_{j=1}^{C_N^M} \overline{p_j(\mathrm{f})} \tag{2}$$

$$= \frac{1}{C_N^M}\frac{1}{M}\sum_{j=1}^{C_N^M}\sum_{ji=1}^{M} p_{ji}(\mathrm{f}) \tag{3}$$

$$= \frac{1}{C_N^M}\frac{1}{M}\frac{M}{N}C_N^M\sum_{i=1}^{N} p_i(\mathrm{f}) \tag{4}$$

$$= \frac{1}{N}\sum_{i=1}^{N} p_i(\mathrm{f}) \tag{5}$$

We get the mathematical expectation of probability density of y_i,

$$\overline{p(\mathrm{f})} = \frac{1}{N}\sum_{i=1}^{N} p_i(\mathrm{f}) \tag{6}$$

It is difficult to compute the density accurately. But this equation shows that there is no relationship between the expectation density $\overline{p(\mathrm{f})}$ and M which means no matter how we choose the rows of the measurement matrix, the expectation density of y_i is unchangeable. So when $\Phi_{n \times n}$ is given, the expectation of the probability density is determined no matter what the sampling order and sampling rate is. Thus we need not to consider the initial sampling condition when we quantize the measurements.

III. PROPOSED METHOD

A. *Setting the Threshold of Discarding*

We can respectively quantize the measurements based on the high frequency part and the low frequency part taking the distribution into consideration. We call them H region and L region. Yet another question should be considered, how to determine the quantization range. Here we set a threshold respectively for each part. If T_h represents the threshold of high frequency and T_l represents the threshold of low frequency, then measurements $|y_h| \le T_h$ and $|y_l - \mu| \le T_l$ will be quantized while other measurements will be discarded directly. Here y_h represents the measurements of high frequency and its expectation is 0, y_l represents the measurements of low frequency, and μ means its mean value. The characteristic of image and the block size will influence the value of μ.

Candès et al demonstrate the conclusion that there is a high probability of exact reconstruction when the sampling ratio M and the K-sparse signal satisfy the relationship as follows,

$$M = O(K\log(K/N)) \tag{7}.$$

So if we consider the quantization error and discarding error, the oversampling rate is easy to be calculated. Simply discarding those measurements which have large quantization errors can get higher reconstruction quality than quantizing them [7]. The difference between previous methods and proposed method is that previous methods quantize all the measurements no matter the measurements are overloaded or not while the proposed method only discards the measurements outside the quantization range. For uniform quantization, with the same number of quantization steps n, the step length of previous methods will be $(\mu + T_l + T_h)/\mathrm{n}$ and proposed method $2(T_l + T_h)/\mathrm{n}$. Fig.2 shows the comparison of proposed and previous methods.

As for uniform quantization, the quantization noise can be represented as：

$$N_q = \sum_{i=1}^{n}\int_{m_{i-1}}^{m_i} (\mathrm{x} - \mathrm{q}_i)^2 f(\mathrm{x})dx \tag{8}$$

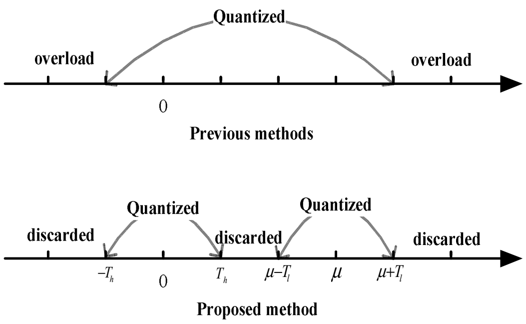

Fig.2 Comparison of the proposed and previous methods

Where n denotes the number of quantization step, m_i represents the threshold of the values which will be quantized to q_i, x is the signal to be quantized, and $f(\mathrm{x})$ is its probability density. Usually n is very large while quantization step is very small.

So we can take $f(\mathrm{x})$ as a constant between m_{i-1} and m_i, and assume the quantization errors are independent. So N_q can be represented as,

$$N_q = \sum_{i=1}^{n} p_i \int_{m_{i-1}}^{m_i} (\mathrm{x}-\mathrm{q}_i)^2 dx \tag{9}$$

$$= \frac{\Delta V^2}{12} \sum_{i=1}^{n} p_i \Delta V \tag{10}$$

$$= \frac{\Delta V^2}{12} \tag{11}$$

We assume that there is no overload, and $\sum_{i=1}^{n} p_i \Delta V$ in above equation equals 1. Then we can get the conclusion as

$$N_q \propto \Delta V^2 \tag{12}$$

So the quantization error using previous methods and proposed method are $(1/12n^2)\times(\mu+T_l+T_h)^2$ and $(1/12n^2)\times 4(T_l+T_h)^2$ respectively. From the analysis we can see that the quantization error of previous methods will be far greater than that of the proposed method when $\mu > T_h + T_l$. If the block size is 32×32, the quantization noise of the latter is about 0.64 times of the former. The quality of the recovery image using the method proposed will be higher than previous methods in theory.

B. *Non-Uniform Quantization*

From section 2, we can see that both high frequency measurements and low frequency measurements satisfy Generalized Gaussian Distribution (GGD). So we design the nonlinear compression function based on the distribution. The normalized compression function can be expressed as following,

$$y = \frac{\ln(1+\delta x)}{\ln(1+\delta)}, 0 \le x \le 1 \tag{13}$$

Fig.3 shows the function when the δ is 1, 5, 10 and 100. So for the *H* region we design the function as,

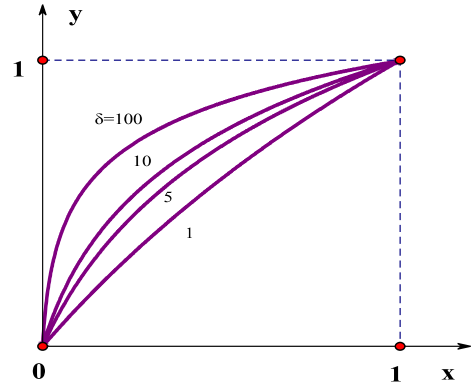

Fig.3 normalized compression function

$$y_h = \pm T_h \frac{\ln(1+\delta |x|/T_h)}{\ln(1+\delta)}, |x| \le T_h \tag{14}$$

And the *L* region we design the function as,

$$y_l = \pm T_l \frac{\ln(1+\delta |x-\mu|/T_l)}{\ln(1+\delta)} + \mu, |x-\mu| \le T_l \tag{15}$$

Above functions can satisfy the two regions and make the steps very small when the measured values are very concentrated. The quantization error will become negligible in such range.

From equation (8), the quantization error of non-uniform will be much smaller than that of uniform. But the error will become larger when the measured values are sparse. So the choice of δ is crucial. The upper and lower bounds of the performance improvement can be represented as,

$$\frac{dy}{dx} = \frac{\delta}{(1+\delta \mathrm{x})\ln(1+\delta)} \tag{16}$$.

If $x \to 0$

$$\frac{dy}{dx} \to \frac{\delta}{\ln(1+\delta)}\Big|_{\delta=5} = 2.8 \tag{17}$$

While if $x \to 1$

$$\frac{dy}{dx} \to \frac{\delta}{(1+\delta)\ln(1+\delta)}\Big|_{\delta=5} = 0.46 \tag{18}$$

IV. PERFORMANCE EVALUATION

A. *Performance at Different Sampling Rates*

Using different sampling rates will get varying numbers of measurements, and if δ changes, the result will also be influenced. Fig.4 shows the peak signal to noise ratio (PSNR) of recovered images when δ is different. Form the fig.4, we can see that when δ is too big or too small, the quality of recovered images is not very ideal. This is because when δ is too big, the quantization error of the measurements far away from the center becomes larger. While the quality of recovered images is best when δ=5, so we set δ=5 in this paper. From equations (17) and (18), we can find that when δ=5, the performance of quantization error can be improved 2.8 times when measurements close to the center, but will drop to 0.46 times of the original if the measurements far away from the center. In order to verify the accuracy of the proposed method, three images of three different sampling rates are used in the paper.

Table I shows the relationship between PSNR and the sampling rates. The recovery algorithm of total-variation minimization (TV) is used. All of the tests use 64 quantization steps. From table I, we can see that PSNR of each image increases with sampling rate.

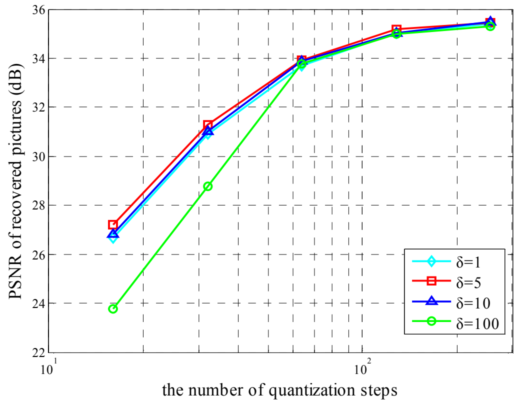

Fig.4 PSNR of different steps and δ

TABLE I. PSNR (dB) OF DIFFERENT SAMPLING RATES, δ=5

Rates \ Image	Lena	Camer-aman	Boat	Goldhill	Peppers
0.3	30.52	29.93	30.75	27.99	31.00
0.4	32.25	31.67	32.45	29.41	32.52
0.5	33.92	33.24	33.93	30.79	33.89

B. Comparison With Previous Method using Different CS reconstruction Algorithms

We compare the proposed method and uniform quantization method at the same sampling rate and δ by using two algorithms of total-variation minimization (TV) and gradient projection for sparse reconstruction (GPSR) to recover the images. And we set the threshold as T_h =200 and T_l = 200 for each region. All the results use 64 quantization steps. Table II shows the reconstruction PSNRs of the two methods.

TABLE II. COMPARISON AT THE RATE OF 0.5, δ=5

Image	TV (dB)		
	Uniform	Proposed	Gain
Lena	32.33	33.93	1.6
Cameraman	31.72	33.59	1.87
Boat	31.82	34.12	2.3
Peppers	32.04	33.89	1.85
Goldhill	29.87	30.79	0.92
Image	GPSR (dB)		
	Uniform	Proposed	Gain
Lena	29.61	30.77	1.16
Cameraman	29.20	30.68	1.48
Boat	28.60	29.66	1.06
Peppers	28.70	29.84	1.14
Goldhill	26.95	27.67	0.72

From table II, we can see that the PSNR of the proposed method is about 1.7dB higher on average than uniform method using TV, and about 1.1 dB higher on average using GPSR. At the same time, we compare the computation time of quantization by using the two methods.

TABLE III COMPUTATION TIME OF QUANTIZATION AT THE RATE 0.5, δ=5

Methods \ Image	Lena	Camer-aman	Boat	Goldhill	Peppers
uniform	0.097s	0.097s	0.097s	0.103s	0.096s
proposed	0.108s	0.101s	0.106s	0.118s	0.101s

From table III, it is shown that the average computation time of uniform and proposed method are 0.098s and 0.107s. The increased time is less than 10 percent using our new method.

The experimental results are based on fixed parameters. When the parameters of image change, δ, T_h and T_l will also need to be re-selected. For example, if the block size changes, the center of the *L* region will change correspondingly, but we can get the similar conclusions. In practical application, for example, distributed compressive sensing networks, the parameters can be set in advance.

V. CONCLUSION

In this paper, a new non-uniform quantization method was proposed based on distribution of CS measurements and coefficients discarding. We proved that the distribution of CS measurements has no relationship with the sampling order and sampling rate. Then we designed the compression function and discarding range for non-uniform quantization. The experiment results showed that the average PSNR of the recovered images using proposed method can be improved by 1.7 dB with TV recovery and 1.1 dB with GPSR recovery in terms of the same parameters.

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