# A New Lattice-Based Adaptive Notch Filtering Algorithm with Improved Mean Update Term

Shinichiro Nakamura, Shunsuke Koshita, Masahide Abe and Masayuki Kawamata
Department of Electronic Engineering, Tohoku University, Sendai, Japan.
E-mail: sinitiro@mk.ecei.tohoku.ac.jp Tel: +81-22-795-7095

***Abstract*—In this paper, we propose a new lattice-based adaptive notch filtering algorithm which has faster convergence characteristics than Regalia's Simplified Lattice Algorithm (SLA). Our algorithm makes use of the weighted sum of SLA and the Lattice Gradient Algorithm. We prove that the mean update term of our algorithm is larger than that of SLA when the input signal consists of a single sinusoid and a background white noise. Furthermore, our algorithm does not change the local convergence characteristics near the sinusoidal frequency. Consequently, the proposed algorithm achieves faster convergence than SLA. A simulation result shows that the proposed algorithm finds the sinusoidal frequency faster than SLA.**

## I. Introduction

Adaptive notch filters are useful when we want to find an unknown frequency of a sinusoid immersed in broadband noise. There are many applications of adaptive notch filters such as communication systems, control systems and howling suppression systems. Many algorithms and approaches for adaptive notch filtering have been developed in the past. For example, the algorithm based on constrained poles and zeros is proposed in [1]. The use of allpass sections is introduced in [2]. In [3] and [4], the normalized lattice structure is applied to the allpass-based adaptive notch filtering. Moreover, Ref. [5] developed a complex adaptive notch filter using the normalized lattice structure. We focus on the Lattice Gradient Algorithm (LGA) [4] and the Simplified Lattice Algorithm (SLA) [3], because they are based on the allpass filter and therefore have some attractive features. These algorithms can achieve unbiased estimation of the frequency even if white background noise is present. Furthermore, they can tune the notch frequency and the notch bandwidth independently.

In order to find the unknown frequency of the sinusoid, LGA tries to minimize the cost function using its gradient. However, this causes the problem that the adaptation becomes very slow when the notch frequency is distant from the sinusoidal frequency. On the other hand, SLA does not use the cost function and improves the convergence speed in this situation. The performance of adaptive notch filtering is dominated by the mean update term, which represents how much the parameter is expected to change. In [4], it is claimed that the mean update term of SLA is always larger than the one of LGA when both have the same local convergence behavior. Therefore, SLA achieves much faster convergence especially in the notch frequencies which are distant from the sinusoidal frequency.

In this paper, we propose a new lattice-based adaptive notch filtering algorithm which has better mean update term than SLA. Our algorithm extends SLA by using the weighted sum of SLA and LGA. We also make a simulation and show the proposed algorithm's superiority over SLA in finding the sinusoidal frequency.

## II. Adaptive Notch Filters Based on Normalized Lattice Structure

Here we consider the situation where the input signal $u(n)$ consists of a single sinusoid $s(n)$ and the additive noise $w(n)$:

$$\begin{aligned} u(n) &= s(n) + w(n) \\ &= A\sin(\omega_s n + \phi) + w(n) \end{aligned} \tag{1}$$

where $A$ is the amplitude, $\omega_s$ is the unknown frequency and $\phi$ is the initial phase of the sinusoid $s(n)$. The initial phase $\phi$ is a random variable uniformly distributed in $[0, 2\pi)$. The additive noise $w(n)$ is the zero-mean white noise.

The notch filter used in LGA and SLA has the following transfer function [3], [4]:

$$H(z) = \frac{1+\sin\theta_2}{2}\frac{1+2\sin\theta_1 z^{-1}+z^{-2}}{1+\sin\theta_1(1+\sin\theta_2)z^{-1}+\sin\theta_2 z^{-2}} \tag{2}$$

where $\theta_1$ is the notch frequency parameter and $\theta_2$ is the notch bandwidth parameter. The notch frequency $\omega_0$ and the 3-dB attenuation bandwidth $\Omega$ are respectively related to $\theta_1$ and $\theta_2$ as

$$\omega_0 = \theta_1 - \pi/2 \tag{3}$$

$$\Omega = 2\tan^{-1}\frac{1-\sin\theta_2}{1+\sin\theta_2}. \tag{4}$$

Equations (3) and (4) show that $\omega_0$ and $\Omega$ depend only on $\theta_1$ and $\theta_2$, respectively. Hence the notch frequency and the notch bandwidth can be independently tuned. This notch filter is based on the normalized lattice structure. The block diagram of this notch filter is shown in Fig. 1.

In order to find the unknown frequency $\omega_s$, LGA and SLA update the parameter $\theta_1$ using the following algorithms [3], [4], respectively:

$$\theta_1(n+1) = \theta_1(n) - \mu\Delta_y(n)y(n) \quad \text{(LGA)} \tag{5}$$

$$\theta_1(n+1) = \theta_1(n) - \mu x_1(n)y(n) \quad \text{(SLA)}. \tag{6}$$

The parameter $\mu$ is the adaptation step size. In (6), $x_1(n)$ is a state variable of the notch filter and SLA uses this to update the notch frequency. In (5), the signal $\Delta_y(n)$ is used in LGA and $\Delta_y(n)y(n)$ is approximately propotional to $\partial E[y^2(n)]/\partial\theta_1(n)$ which is the gradient of the cost function. The signal $\Delta_y(n)$ is obtained by filtering $x_1(n)$ using the bandpass filter $G(z) = 1 - H(z)$ which is shown in Fig. 2.

We next address the mean update term [4], [5] which represents how much the parameter is changed by the algorithms in mean. The mean update term determines the convergence behavior of an adaptive notch filtering algorithm. The mean update terms of LGA and SLA are defined by

$$f_{\mathrm{LGA}}(\omega_0) \equiv -E[\Delta_y(n)y(n)] \quad (\mathrm{LGA}) \tag{7}$$

$$f_{\mathrm{SLA}}(\omega_0) \equiv -E[x_1(n)y(n)] \quad (\mathrm{SLA}) \tag{8}$$

respectively. By using Parseval's theorem, the mean update terms can be expressed as follows:

$$\begin{aligned} f_{\mathrm{LGA}}(\omega_0) &= -E[\Delta_y(n)y(n)] \\ &= -\langle F(z)G(z), H(z)\rangle_S \end{aligned} \tag{9}$$

$$\begin{aligned} f_{\mathrm{SLA}}(\omega_0) &= -E[x_1(n)y(n)] \\ &= -\langle F(z), H(z)\rangle_S \end{aligned} \tag{10}$$

where $F(z)$ is the transfer function from the input $u(n)$ to $x_1(n)$:

$$F(z) = \frac{\cos\theta_1\cos\theta_2 z^{-1}}{1+\sin\theta_1(1+\sin\theta_2)z^{-1}+\sin\theta_2 z^{-2}}. \tag{11}$$

In addition, $\langle H_1(z), H_2(z)\rangle_S$ denotes the inner product of two functions $H_1(z)$ and $H_2(z)$ induced by the input signal $u(n)$ [4] defined as

$$\langle H_1(z), H_2(z)\rangle_S \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} H_1(e^{j\omega})H_2^*(e^{j\omega})S(\omega)d\omega \tag{12}$$

where $S(\omega)$ is the power spectral density function of the input signal $u(n)$:

$$S(\omega) = \frac{\pi A^2}{2}[\delta(\omega-\omega_s)+\delta(\omega+\omega_s)] + \sigma^2. \tag{13}$$

Here $\sigma^2$ is the variance of the white noise. Figure 3 shows the mean update terms of LGA and SLA. This figure shows that SLA always has larger mean update term than LGA for the same local convergence properties:

$$\left.\frac{\partial f_{\mathrm{LGA}}(\omega_0)}{\partial\omega_0}\right|_{\omega_0=\omega_s} = \left.\frac{\partial f_{\mathrm{SLA}}(\omega_0)}{\partial\omega_0}\right|_{\omega_0=\omega_s}. \tag{14}$$

Therefore, SLA has better convergence property than LGA.

## III. Proposed Method

In this section, we propose a new adaptive notch filtering algorithm which has better mean update term than SLA. The block diagram of our algorithm is shown in Fig. 4 and the adaptation algorithm is given by

$$\theta_1(n+1) = \theta_1(n) - \mu\psi(n)y(n) \tag{15}$$

where $\mu$ is the adaptation step size and $y(n)$ is the notch output. We give the signal $\psi(n)$ as follows:

$$\psi(n) = kx_1(n) + (1-k)\Delta_y(n) \tag{16}$$

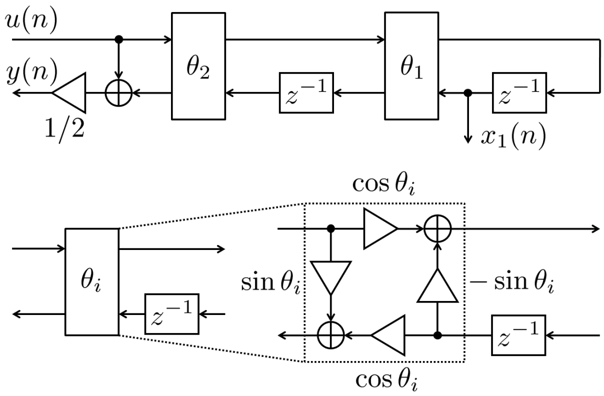


Fig. 1. Block diagram of the notch filter $H(z)$ based on the normalized lattice structure.

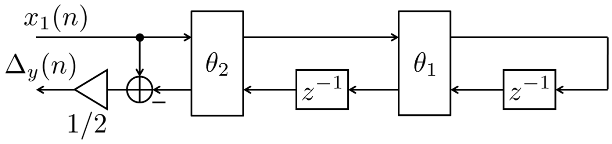


Fig. 2. Block diagram of the bandpass filter $G(z)$ which is used in LGA.

where $k$ is a constant, $x_1(n)$ is the state variable used in SLA and $\Delta_y(n)$ is the signal used in LGA. Therefore the proposed algorithm is a weighted sum of LGA and SLA. As special cases, the proposed algorithm is equivalent to LGA and SLA if $k = 0$ and $k = 1$, respectively. However, we set the parameter $k$ to be $k > 1$ in our proposed algorithm.

We next prove the facts that our algorithm has better convergence property than SLA and the more improvement of the convergence property is achieved when larger $k$ is used. In order to show these facts, we analyze the mean update term. The mean update term of our proposed algorithm is defined as follows:

$$f_{\mathrm{prop}}(\omega_0) \equiv -E[\psi(n)y(n)]. \tag{17}$$

From (16) and (17), we obtain the following relationships:

$$\begin{aligned} f_{\mathrm{prop}}(\omega_0) &= kf_{\mathrm{SLA}}(\omega_0) + (1-k)f_{\mathrm{LGA}}(\omega_0) \\ &= f_{\mathrm{SLA}}(\omega_0) + (k-1)f_{\mathrm{diff}}(\omega_0) \end{aligned} \tag{18}$$

where $f_{\mathrm{diff}}(\omega_0) \equiv f_{\mathrm{SLA}}(\omega_0) - f_{\mathrm{LGA}}(\omega_0)$. In the proposed algorithm, the component $f_{\mathrm{diff}}(\omega_0)$ plays an important role to improve the mean update term. For $k > 1$, the term $f_{\mathrm{diff}}(\omega_0)$ is increased and this improves the mean update term without effect on the local convergence property at $\omega_0 = \omega_s$. This is because $f_{\mathrm{diff}}(\omega_0)$ has zero gradient at $\omega_0 = \omega_s$ and $f_{\mathrm{diff}}(\omega_0)$ has the same sign as $f_{\mathrm{SLA}}(\omega_0)$. We will prove them in the

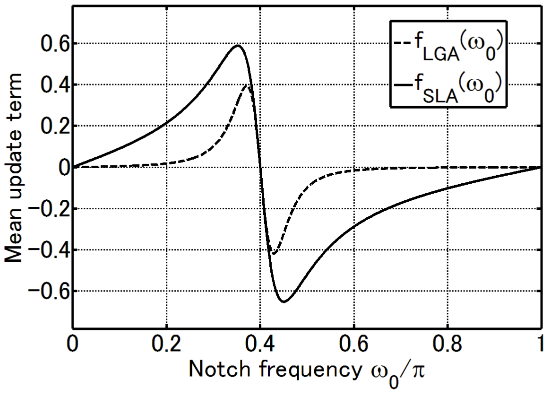


Fig. 3. Mean update terms of LGA and SLA with $\omega_s = 0.4\pi$, $\Omega = 0.1\pi$ and $A = 1.0$.

following. $f_{\rm diff}(\omega_0)$ is calculated as below:

$$\begin{aligned} f_{\rm diff}(\omega_0) &= f_{\rm SLA}(\omega_0) - f_{\rm LGA}(\omega_0) \\ &= \langle F(z)G(z), H(z)\rangle_S - \langle F(z), H(z)\rangle_S \\ &= \langle F(z)[1 - H(z)], H(z)\rangle_S - \langle F(z), H(z)\rangle_S \\ &= -\langle F(z)H(z), H(z)\rangle_S \\ &= -\frac{A^2}{2}|H(e^{j\omega_s})|^2 \mathrm{Re}[F(e^{j\omega_s})]. \end{aligned} \tag{19}$$

Here it follows that

$$-\frac{A^2}{2}\mathrm{Re}[F(e^{j\omega_s})] = f_{\rm SLA}(\omega_0) \tag{20}$$

because

$$\mathrm{Re}[F(e^{j\omega_s})] = (1 + \sin\theta_2)\cos\theta_1\cos\theta_2 \frac{\cos\omega_s + \sin\theta_1}{|D(e^{j\omega_s})|^2} \tag{21}$$

$$f_{\rm SLA}(\omega_0) = -\frac{A^2}{2}(1 + \sin\theta_2)\cos\theta_1\cos\theta_2 \frac{\cos\omega_s + \sin\theta_1}{|D(e^{j\omega_s})|^2} \tag{22}$$

where $D(z)$ is the denominator polynomial of $H(z)$:

$$D(z) = 1 + \sin\theta_1(1 + \sin\theta_2)z^{-1} + \sin\theta_2 z^{-2}. \tag{23}$$

Therefore, from (19) and (20), we obtain the simple expression on $f_{\rm diff}(\omega_0)$:

$$f_{\rm diff}(\omega_0) = |H(e^{j\omega_s})|^2 f_{\rm SLA}(\omega_0) \tag{24}$$

and its derivative at $\omega_0 = \omega_s$ becomes

$$\begin{aligned} \left.\frac{\partial f_{\rm diff}(\omega_0)}{\partial\omega_0}\right|_{\omega_0=\omega_s} &= \left.\frac{\partial|H(e^{j\omega_s})|^2}{\partial\omega_0}\right|_{\omega_0=\omega_s} f_{\rm SLA}(\omega_s) \\ &\quad + |H(e^{j\omega_s})|^2 \left.\frac{\partial f_{\rm SLA}(\omega_0)}{\partial\omega_0}\right|_{\omega_0=\omega_s} \\ &= 0\cdot 0 + 0\cdot \left.\frac{\partial f_{\rm SLA}(\omega_0)}{\partial\omega_0}\right|_{\omega_0=\omega_s} \\ &= 0. \end{aligned} \tag{25}$$

This means that increasing the component $f_{\rm diff}(\omega_0)$ does not change the local convergence behavior. Additionally, from the fact that $|H(e^{j\omega_s})|^2 \geq 0$ and (24), we can show that $f_{\rm diff}(\omega_0)$ and $f_{\rm SLA}(\omega_0)$ have the same sign:

$$\frac{f_{\rm diff}(\omega_0)}{f_{\rm SLA}(\omega_0)} \geq 0. \tag{26}$$

This proves that the proposed algorithm has better mean update term than SLA in all $\omega_0$ by setting $k$ to be $k > 1$. Figure 5 shows the mean update term of the proposed algorithm. From this figure, we see that the proposed algorithm has larger mean update term than SLA for the same local convergence property. In addition, the mean update term becomes larger as $k$ increases.

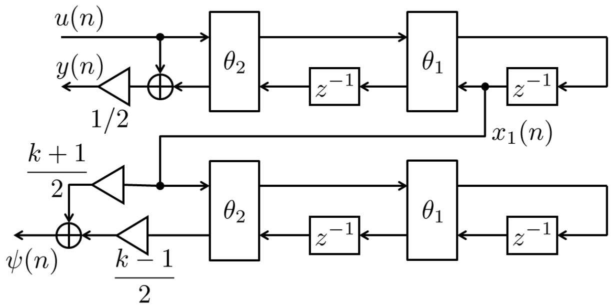


Fig. 4. Block diagram of the proposed algorithm.

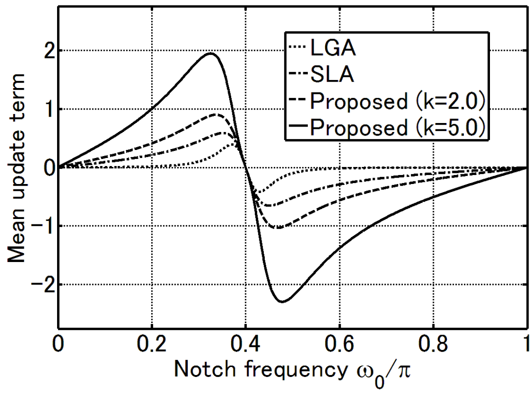


Fig. 5. Mean update terms of the proposed algorithm for $\omega_s = 0.4\pi$, $\Omega = 0.1\pi$ and $A = 1.0$.

## IV. Simulation Example

Figure 6 shows the frequency estimate with the proposed algorithm and SLA. In this simulation we used the sinusoid with the amplitude $A = 1.0$ and the white Gaussian noise of which variance is adjusted to have a signal-to-noise ratio of 0 dB. The unknown frequency $\omega_0$ of the sinusoid changes suddenly every 5000 samples. The bandwidth of the notch filter is $\Omega = 0.1\pi$ and the step-size parameter is $\mu = 0.004$. The parameter $k$ for the proposed algorithm is set to be $k = 2.0$. From Fig. 6 we see that the proposed algorithm finds the unknown frequency faster than SLA. As for the local convergence property at $\omega_0 = \omega_s$, we see the same estimation errors both in the proposed algorithm and SLA.

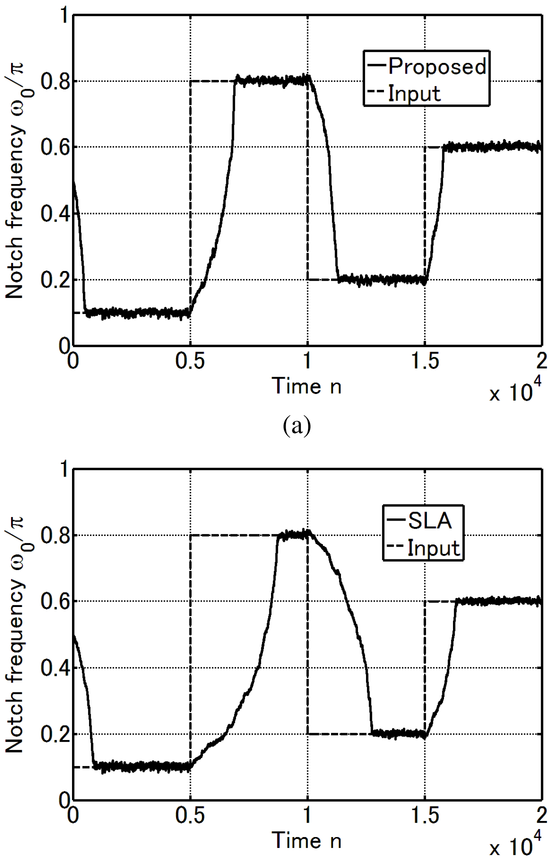


Fig. 6. Frequency estimates for the two adaptive notch filtering algorithms: (a) the proposed algorithm and (b) SLA.

It should be noted that, for too large $k$, the mean update term becomes too large and it causes a problem: in such a case, we experimentally confirmed that the local convergence property of our proposed algorithm is different form that of SLA. Thus the analysis for the upper bound of $k$ that keeps the local convergence property is one of our future tasks.

## Acknowledgment

This work was supported in part by the Casio Science Promotion Foundation.

## V. Conclusions

We developed a new lattice-based adaptive notch filtering algorithm which has better mean update term than SLA. In the proposed algorithm, we improved the mean update term by using the weighted sum of LGA and SLA. The simulation result showed that the proposed algorithm finds the unknown frequency faster than SLA. Moreover, the convergence speed of the notch frequency is increased when the parameter $k$ is set to be large.

## References

[1] A. Nehorai, "A minimal parameter adaptive notch filter with constrained poles and zero," *IEEE Trans. Signal Processing*, vol. 33, no. 4, pp. 983–996, Aug. 1985.

[2] J. A. Chambers and A. G. Constantinides, "Frequency tracking using constrained adaptive notch filters synthesised from allpass sections," *IEEE Proc. Radar and Signal Processing*, vol. 137, no. 6, pp. 475–481, Dec. 1990.

[3] P. A. Regalia, "An improved lattice-based adaptive IIR notch filter," *IEEE Trans. Signal Processing*, vol. 39, no. 9, pp. 2124–2128, Sep. 1991.

[4] P. A. Regalia, *Adaptive IIR Filtering in Signal Processing and Control*, Marcel Dekker, 1995.

[5] P. A. Regalia, "A complex adaptive notch filter," *IEEE Signal Processing Lett.*, vol. 17, no. 11, pp. 937–940, Nov. 2010.