# Generalized Polynomial Wigner Spectrogram for High-Resolution Time-Frequency Analysis

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***Abstract*— A good time-frequency (TF) analysis method should have the advantages of high clarity and no cross term. However, there is always a trade-off between the two goals. In this paper, we propose a new TF analysis method, which is called the generalized polynomial Wigner spectrogram (GPWS). It combines the generalized spectrogram (GS) and the polynomial Wigner-Ville distribution (PWVD). The PWVD has a good performance for analyzing the instantaneous frequency of a high order exponential function. However, it has the cross term problem in the multiple component case. By contrast, the GS can avoid the cross term problem, but its clarity is not enough. The proposed GPWS can combine the advantages of the PWVD and the GS. It can achieve the goals of high clarity, no cross term, and less computation time simultaneously. We also perform simulations to show that the proposed GPWS has better resolution than other TF analysis methods.**

## I. INTRODUCTION

Time-frequency (TF) analysis [1-17] is useful for adaptive signal processing. With it, one can observe the signal variation in both the time and the frequency domains and estimate the instantaneous frequency of the signal. Some popular TF analysis methods are described as follows.

The well-known short-time Fourier transform (STFT) [1-3] is defined as:

$$STFT_x(t,f)=\int_{-\infty}^{\infty} w(t-\tau)e^{-j2\pi f\tau}x(\tau)d\tau . \qquad (1)$$

It is the simplest T-F analysis method. Specially, when $w(t)$ is a Gaussian function, it becomes the Gabor transform [4]

$$G_x(t,f)=\sqrt[4]{\sigma}\int_{-\infty}^{\infty} e^{-\sigma\pi(t-\tau)^2}e^{-j2\pi f\tau}x(\tau)d\tau \qquad (2)$$

where $\sigma$ is the standard deviation of the Gaussian window. The STFT is easy to implement. With the STFT, the instantaneous frequency of a signal can be observed and there is no cross term problem. However, the TF map of the STFT has poor clarity. Furthermore, due to the Heisenberg uncertainty principle [5], it is hard to make the TF map of the STFT have high resolutions in both the time domain and the frequency domain simultaneously. For example, a broader window size will make the frequency resolution higher but causes a worse time resolution.

To solve the trade-off problem the generalized spectrogram is proposed [6][7]. The conventional spectrogram is:

$$SP_x(t,f)=|STFT_x(t,f)|^2. \qquad (3)$$

It is the square of the STFT magnitude. In [6][7], it is generalized into the generalized spectrogram:

$$SP_x(t,f)=\left|STFT_{x,W_1}(t,f)\right|\left|STFT_{x,W_2}(t,f)\right| \qquad (4)$$

where $STFT_{x,W_1}(t,f)$ means the STFT using the window of $W_1(t)$. Since the STFT has a good time resolution when the window is narrower and has a good frequency resolution when the window is wider, if one chooses $W_1(t)$ as a narrower window and $W_2(t)$ as a wider window, then $SP_x(t,f)$ has higher resolutions in both the time and the frequency domain.

Although the TF map of the generalized spectrogram is clearer than that of the STFT, some TF analysis methods, such as the Wigner distribution function and Cohen's class distributions, have even higher clarity than the generalized spectrogram.

The Wigner distribution function (WDF) [8][9] is defined as:

$$W_x(t,f)=\int_{-\infty}^{\infty} x(t+\frac{\tau}{2})x^*(t-\frac{\tau}{2})e^{-j2\pi\tau f}d\tau . \qquad (5)$$

The WDF has many good mathematical properties and its TF map has very high clarity. However, it suffers from severe cross term problems. If $x(t)$ consists of multiple components or a higher order phase component, then, due to the autocorrelation operation, the cross term occurs.

Cohen's class distribution [10][11] is a generalization of the WDF:

$$C_x(t,f)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_x(\eta,\tau)\Phi(\eta,\tau)e^{j2\pi(\eta t-\tau f)}d\eta d\tau \qquad (6)$$

where $A_x(\eta,\tau)$ is the ambiguity function:

$$A_x(\eta,\tau)=\int_{-\infty}^{\infty} x(t+\frac{\tau}{2})x^*(t-\frac{\tau}{2})e^{-j2\pi\eta t}dt . \qquad (7)$$

The clarity of Cohen's class distribution is near to that of the WDF. Moreover, if $\Phi(\eta,\tau)$ is a lowpass mask, the cross term can be attenuated. However, Cohen's class distribution requires a lot of computation time. Moreover, there is a trade-off about designing the pass region of $\Phi(\eta,\tau)$. If the pass region is narrow, more cross terms can be removed but the clarities of auto terms are reduced. If the pass region is width, auto terms have higher clarity but the ability of cross term removing is reduced.

The polynomial Wigner-Ville distribution (PWVD) [12-14] is also a generalization of the WDF. The $q^{th}$ order PWVD is defined as

$$PWVD_x(t,f)=\int_{-\infty}^{\infty}\left[\prod_{l=1}^{q/2} x(t+d_l\tau)x^*(t-d_{-l}\tau)\right]e^{-j2\pi\tau f}d\tau \quad (8)$$

where $q$ should be an even integer. When $q = 2$ and $d_1 = d_{-1} = 0.5$, it becomes the WDF. The PWVD can effectively reduce the cross term for the signal with a higher order phase. That is, if the input signal has the form of

$$C\cdot\exp[j(a_k t^k + a_{k-1} t^{k-1} + \ldots + a_0)], \tag{9}$$

then, using the $(2k)^{\text{th}}$-order PWVD, the cross term can be removed. However, as the simulation in Fig. 1(d), the PWVD cannot remove the cross term in the multiple component case.

In Table I, we make a summary for the TF analysis methods, their clarities, and whether they can avoid cross-term problems.

In this paper, we propose a new time frequency analysis method, which is called the generalized polynomial Wigner spectrogram (GPWS). Its TF map has very high clarity along both the time and the frequency axes. Moreover, it can avoid the cross term problem in both the higher order phase case and the multiple component case. Simulations show that the proposed GPWS has better performances than existing methods for analyzing the time-frequency distributions of signals.

## II. Generalized Polynomial Wigner Spectrogram

### A. Definition

The existing TF analysis methods can be classified into two types. The first type is the windowed spectrum analysis method, which includes the STFT and the generalized spectrogram. These methods are based on the Fourier transform (FT) of the input signal multiplied by a window. The second type is the quadratic TF analysis method, which includes the WDF, Cohen's class distribution, and the PWVD. These methods are based on the FT of auto correlation functions [9]. As in Table I, the windowed spectrum analysis method can avoid the cross term and the quadratic TF analysis method has very high clarity.

Among windowed spectrum analysis methods, the generalized spectrogram has the best performance, since it has acceptable clarities along both $t$-axis and $f$-axis. Among quadratic TF analysis methods, the PWVD has the best performance, since it can well analyze the TF distribution of a signal even if it has a higher order phase. Therefore, we propose a new class of TF analysis methods, which is a hybrid of the generalized spectrogram and the PWDF. We call it the *generalized polynomial Wigner spectrogram (GPWS)*:

$$C_x(t,f) = p(SP_x(t,f), |PWVD_x(t,f)|) \tag{10}$$

where $p(x, y)$ is any function with two variables and $SP_x(t, f)$ and $PWVD_x(t,f)$ are defined as in (4) and (8), respectively.

If $p(x, y)$ in (10) is chosen properly such that

$$p(x,y) = 0 \quad \text{when } \min(x,y) = 0, \tag{11}$$

then the resultant GPWS can effectively avoid the cross-term problem while maintaining the clarity as good as that of the PWVD. For example, one can choose $p(x, y) = xy$. Then

$$C_x(t,f) = SP_x(t,f)\left|PWVD_x(t,f)\right|. \tag{12}$$

More general, $p(x, y)$ can be chosen as $x^{\alpha}y^{\beta}$. Then the GPWS becomes

$$C_x(t,f) = SP_x^{\alpha}(t,f)\left|PWVD_x^{\beta}(t,f)\right|. \tag{13}$$

Alternatively, we can add a thresholding operation and define the GPWS as:

TABLE I
SUMMARY OF TIME-FREQUENCY (TF) ANALYSIS METHODS. WE USE 'O', 'Δ', AND '×' TO SHOW WHETHER A TF ANALYSIS METHOD HAS THE ADVANTAGES OF HIGH CLARITIES AND AVOIDING CROSS TERMS.

| TF analysis Methods | High clarity along $t$-axis | High clarity along $f$-axis | Avoiding the cross-term in the high-order phase case | Avoiding the cross-term in the multiple component case |
|---|---|---|---|---|
| STFT (narrow window) | Δ | × | O | O |
| STFT (wider window) | × | Δ | O | O |
| Generalized Spectrogram | Δ | Δ | O | O |
| Wigner distribution | O | O | × | × |
| Cohen's class distribution | O | O | Δ | Δ |
| PWVD | O | O | O | × |
| **Proposed GPWS** | O | O | O | O |

$$C_x(t,f) = thr[SP_x^{\alpha}(t,f)]\left|PWVD_x^{\beta}(t,f)\right| \tag{14}$$

where

$$thr(x) = x - \Delta \quad \text{if } x > \Delta, \quad thr(x) = 0 \quad \text{if } x \le \Delta. \tag{15}$$

Note that, in (12)-(14), the multiplication operation is applied. If $Z = XY$, then $Z$ is nonzero only when both $X$ and $Y$ are nonzero. Therefore, if $PWVD_x(t, f)$ has a cross term at $(t_1, f_1)$, although $PWVD_x(t_1, f_1)$ is nonzero, since $SP_x(t_1, f_1) = 0$, $C_x(t_1, f_1)$ is also equal to 0 and the cross term problem can be solved. On the other hand, due to the limitation of clarity, $SP_x(t_2, f_2)$ may be nonzero if $f_2$ is near to but not equal to the instantaneous frequency at $t = t_2$. However, since the PWVD has a very high clarity and $PWVD_x(t_2, f_2) \cong 0$, $C_x(t_2, f_2)$ is also near to zero. Therefore, the proposed GPWS can both avoid the cross term problem and have a very high clarity.

Note that one can also use the logic operation and the minimum operation to define the GPWS. For example

$$C_x(t,f) = \min\{A_1\, SP_x(t,f), A_2\left|PWVD_x(t,f)\right|\} \tag{16}$$

or

$$C_x(t,f) = SP_x^{\alpha}(t,f)PWVD_x^{\beta}(t,f)\cdot\{[SP_x(t,f) > \Delta_1] \;\&\left[\left|PWVD_x(t,f)\right| > \Delta_2\right]\}. \tag{17}$$

### B. Implementation

It seems that the proposed GPWS is a hybrid of the generalized spectrogram and the PWVD and hence the computation loading is increased. In fact, the proposed GPWS can be implemented in very efficient ways.

Note that, when computing $Z = XY$, if $X = 0$, then no matter what the value of $Y$ is, $Z$ is equal to zero. Therefore, if the generalized spectrogram of a signal is zero or near to zero at a location, then we do not have to calculate the PWVD of the signal at this location.

For example, if the GPWS is defined as in (14), we can first compute the generalized spectrogram $SP_x(t, f)$. Then we only have to compute the PWVD in the location where $SP_x(t, f) > \Delta$. Its implementation process is as follows:

(Step 1) First, we compute the STFT of $x(t)$ using a narrow window, such as the Gaussian function with a large value of $\sigma_1$:

$$G_{x,\sigma_1}(t,f) = \sqrt[4]{\sigma_1}\int_{-\infty}^{\infty} e^{-\sigma_1\pi(t-\tau)^2} e^{-j2\pi f\tau} x(\tau)\,d\tau . \quad (18)$$

Note that the support of $x(t)\exp[-\sigma_1\pi(t-\tau)^2]$ is very narrow when $\sigma_1$ is large. The FT of a narrow support signal can be implemented by many efficient algorithms, such as the pruned fast Fourier transform (pruned FFT) algorithm [18][19].

(Step 2) Then, we calculate the STFT of the input signal $x(t)$ using a wider window:

$$G_{x,\sigma_2}(t,f) = \sqrt[4]{\sigma_2}\int_{-\infty}^{\infty} e^{-\sigma_2\pi(t-\tau)^2} e^{-j2\pi f\tau} x(\tau)\,d\tau \quad (19)$$

where $\sigma_2$ is a small value to make the window wider. Since

$$G_{x,\sigma_2}(t,f) = e^{-j2\pi t f} G_{X,1/\sigma_2}(f,-t) \quad (20)$$

where

$$G_{X,1/\sigma_2}(t,f) = \sqrt[4]{\sigma_2^{-1}}\int_{-\infty}^{\infty} e^{-\pi(t-\tau)^2/\sigma_2} e^{-j2\pi f\tau} X(\tau)\,d\tau \quad (21)$$

and $X(f) = FT[x(t)]$. Therefore, to compute (19), we can first compute $X(f)$. Then, since the support of $X(\tau)\exp[-\pi(t-\tau)^2/\sigma_2]$ is narrow, one can also use the efficient pruned FFT algorithm [18][19] to compute (21). Therefore, as Step 1, Step 2 can also be implemented in a very efficient way.

(Step 3) Then, use (4) to compute the generalized spectrogram from and use (15) to find the value of $thr[SP_x^{\alpha}(t,f)]$.

(Step 4) Then, we calculate $PWVD_x(t, f)$. Since $PWVD_x(t, f)$ should be calculated only when $thr[SP_x^{\alpha}(t,f)] \neq 0$, this step can again be implemented efficiently by the pruned FFT algorithm.

(Step 5) After multiplying $|PWVD_x(t, f)|$ by $thr[SP_x^{\alpha}(t,f)]$, the GPWS of the input signal is obtained.

## III. SIMULATIONS

In this section, we compare the performances of the proposed GPWS and the existing TF analysis methods.

In Fig. 1, the input is a real signal that has a cubic phase:

$$x(t) = 2\cos((t-5)^3 + 4\pi t) . \quad (22)$$

It can be viewed as a combination of two exponential components, i.e., $x(t) = \exp[j(t-5)^3 + j4\pi t] + \exp[-j(t-5)^3 - j4\pi t]$. In Fig. 2, the input is an exponential function whose phase is a 4th order polynomial:

$$x(t) = \exp(j(t-5)^4 - j5\pi(t-5)^2) . \quad (23)$$

From Figs. 1 and 2, one can see that the proposed GPWS has better TF analysis performance than other TF analysis methods. The Gabor transform has limited clarity. The WDF and Cohen's class distribution have obvious cross-term problems. The PWVD has high clarity, but it suffers from the cross term problem especially in the multiple component case. The generalized spectrogram has no cross term problem, but its clarity is not high enough. When using the proposed GPWS, from Figs. 1(f) and 2(f), we can see that the TF maps are very clear and there is no cross term problem.

## IV. CONCLUSION

In this paper, a very effective TF analysis method, which is called the generalized polynomial Wigner spectrogram (GPWS), is proposed. It combines the advantages of the windowed spectrum analysis method and the quadratic TF analysis methods and can achieve all of the four goals listed in Table I (high clarities along both $t$-axis and $f$-axis and no cross term in both the higher order phase case and the multiple component case). Moreover, with the pruned FFT algorithm, it can be implemented in a very efficient way. Simulations show that the proposed GPWS can precisely analyze the TF distribution of time-variant signals and multiple component signals.

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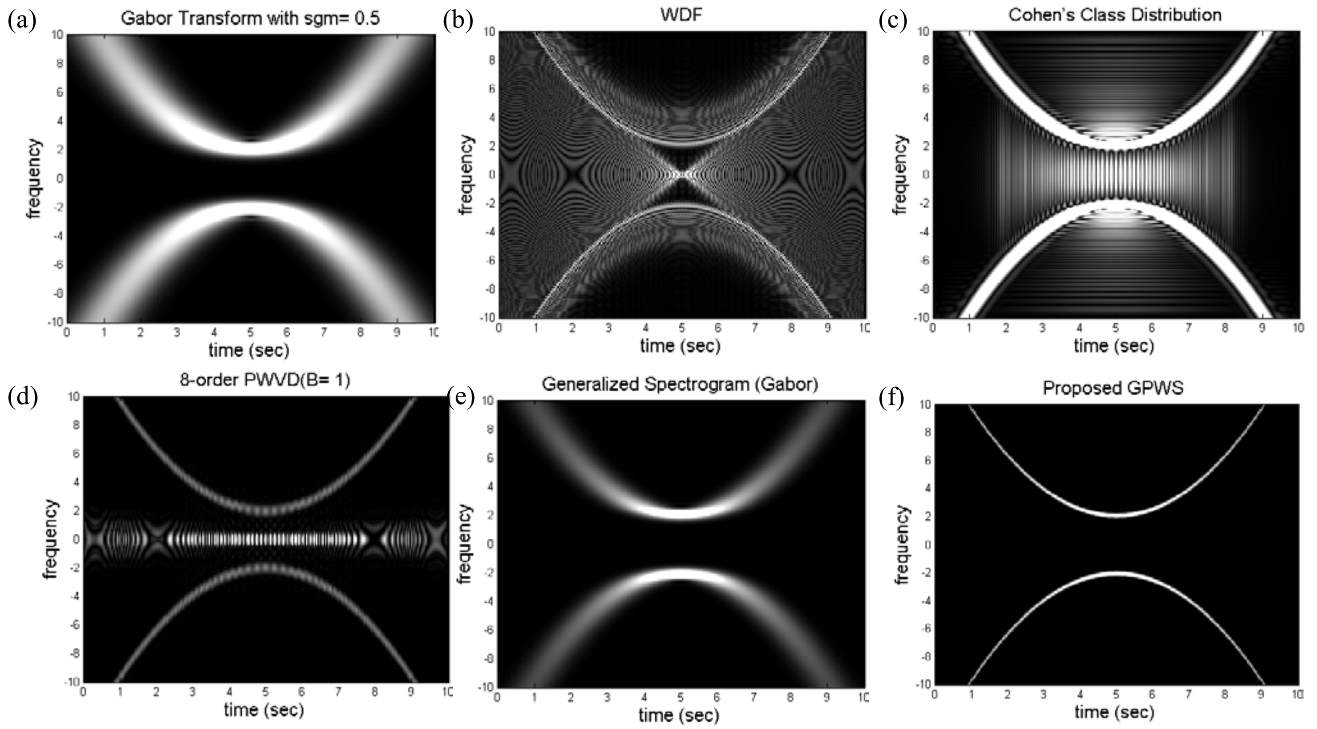


Fig. 1 Time-frequency analysis for $x(t) = 2\cos((t-5)^3 + 4\pi t)$. (a) GT with $\sigma$ =0.5. (b) WDF. (c) Cohen's class distribution. (d) 8th-order PWVD. (e) Generalized spectrogram. (f) Proposed GPWS.

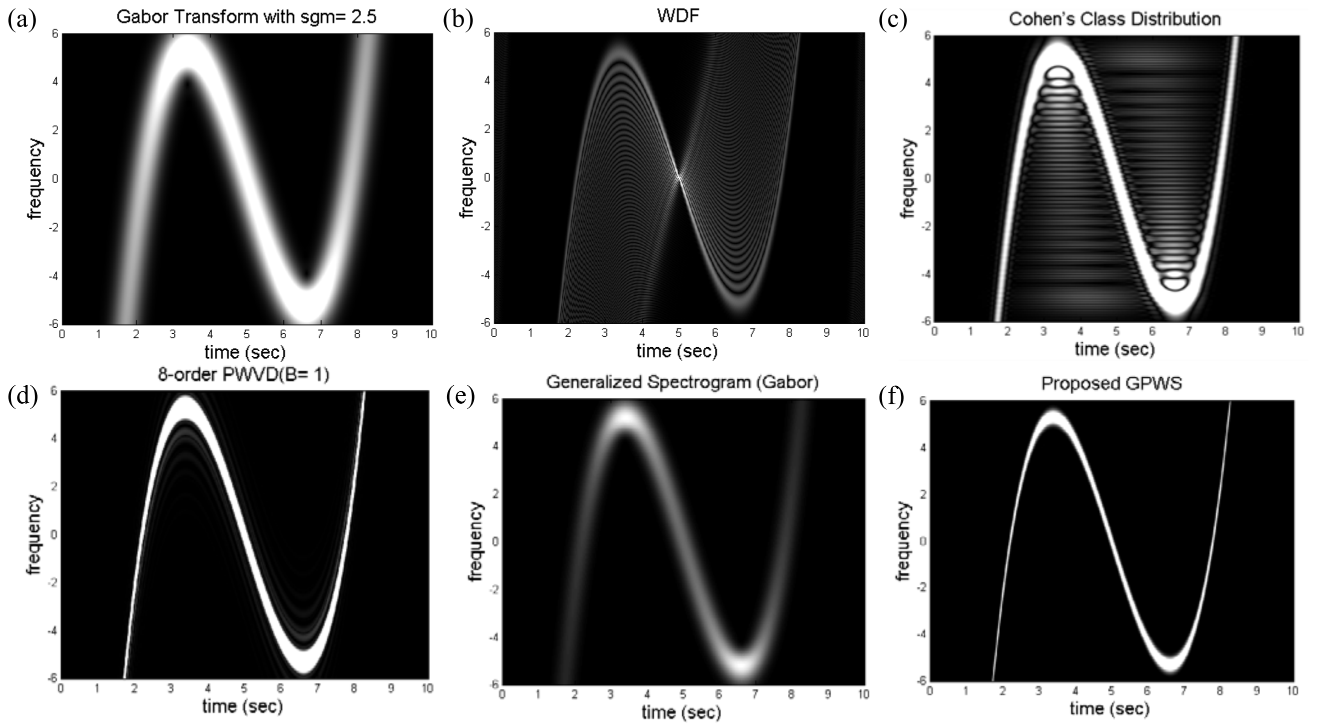


Fig. 2 Time-frequency analysis for a signal whose phase is a 4th order polynomial. $x(t) = \exp(j(t-5)^4 - j5\pi(t-5)^2)$. (a) GT with $\sigma$ = 2.5. (b) WDF. (c) Cohen's class distribution. (d) 8th-order PWVD. (e) Generalized spectrogram. (f) Proposed GPWS.