Orthogonal Random Projection Based Tensor Completion for Image Recovery

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***Abstract*—Thanks to the multi-linearity nature of data, tensor completion approaches often achieve significantly improved performance than matrix based techniques. These methods mostly use the Tucker model and need to frequently compute the singular value decompositions (SVD) of unfolding matrices, hence are not qualified for large-scale data. In this paper, a randomized tensor completion method is proposed to solve this problem. In the proposed method, efficient orthogonal random projection is employed to take the place of SVD, which significantly reduce the computational complexity. Extensive experimental results on color image recovery applications showed that the proposed method is considerably faster than state-of-the-art while achieving comparable peak signal-to-noise ratio.**

I. Introduction

Low-rank matrix completion problems have been studied extensively in the past decades [1] [2] [3] [4] [5]. However, these methods are not competent for higher-order data, such as color images and videos. These kinds of higher-order data could be represented by tensors without destroying their natural structure. Therefore, tensor based methods have been applied to many applications [6], such as computer vision [7] [8] [9], data mining [10], collaborative filtering [11], as well as color image and video recovery [12]. The color image and video recovery problems could be reviewed as tensor completion problems. In [7], Liu *et al* mentioned that the core problem of the tensor completion lies on how to built up the relationship between known elements and the unknown ones, it also could be viewed as the low-rank approximation problem of tensors. The low-rank tensor approximation problem can be formulated as follows:

$$\min_{\mathcal{C}} \ \text{rank}(\mathcal{C}), \quad \text{s.t. } \mathcal{P}_{\Omega}(\mathcal{C} - \mathcal{M}) = 0, \tag{1}$$

where $\mathcal{C}$, the tensor for estimation which is supposed to low-rank, and $\mathcal{M}$ is the tensor with missing values, Ω is an index set denoting the indices of observations, $\mathcal{P}$ is a linear operator, and $\mathcal{P}_{\Omega}$ means that extracts the elements which indices are in the Ω set.

As the rank$(\cdot)$ function is a non-convex, Candès and Recht [2], Recht *et al.* [1], and Candès and Tao [5] proved that, under certain conditions, the rank minimization problem of tensors could be well approximated by a convex optimization problem.

There are many tensor completion methods, such as low-rank tensor completion by tensor-nuclear-norm (TNN) [13] [14], the simple low rank tensor completion (SiLRTC) algorithm. All these methods need to frequently compute singular value decompositions (SVD) [15] [6], and they are not competent for large-scale data since the cost of SVD is very high ($O(m^2n + n^3)$ for a $m \times n$ matrix). In order to solve this problem, we propose a novel low-rank orthogonal random projection method for tensor completion. Our method can avoid the SVD operations, hence is extremely fast and very suitable for large-scale data processing. There are many methods also use the randomized algorithm to solve this computational problems, such as [16], has propose a fast and accurate approximation method method for SVT, [17] has propose a fast and randomized tensor CP decomposition algorithm based on sketching, and [18] has propose a MACH and randomized tensor Tucker decomposition, but they do not really avoid SVD operations.

The rest of the paper is organized as follows. Basic notations, symbols, and some preliminaries are provided in section 2, the formulation of tensor completion introduction in section 3, the new algorithm is detailed in section 4. Experimental results are presented in section 5, and conclusions are made in section 6.

II. Notation and Preliminaries

In this section, we provide the explanation of the notations and introduce some tensor operations that appear in this paper.

A. Notation

Following [19], we use boldface lowercase letters to denote the vectors, e.g., $\mathbf{a}$, the boldface capital letters for matrices, e.g., $\mathbf{A}$, the Euler script letters for higher-order tensors, e.g., $\mathcal{A}$, lowercase letters for scalars, e.g., a. The N-mode tensor is denoted by $\mathcal{A} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$, where I_j is the dimensionality of j-th mode.

B. matricization for tensor

Unfolding or flattening a tensor is also known as matricization of a tensor. For a higher-order tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$, its n-th mode unfolding is denoted by $\mathbf{X}_n \in \mathbb{R}^{I_n \times I_1 I_2 \cdots I_{n-1} I_{n+1} \cdots I_N}$. Fig. 1 displays the n-mode unfolding for a 3rd-order tensor with Kolda.

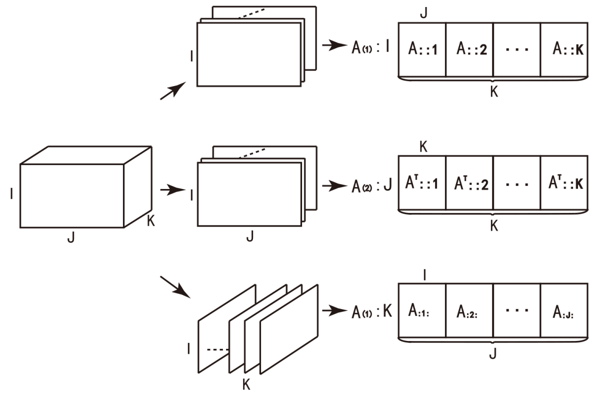

Fig. 1. Matricization of a 3rd-order tensor

C. The n-mode product of tensor

A tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ multiply by a matrix $\mathbf{A}_n \in \mathbb{R}^{J_n \times I_n}$ can be expressed as $\mathcal{X} \times_n \mathbf{A}_n$ and its size is $I_1 \times I_2 \times \cdots \times I_{n-1} \times J_n \times I_{n+1} \times \cdots \times I_N$. The elements of this tensor can be written as follows:

$$(\mathcal{X} \times_n \mathbf{A}_n)_{i_1 \cdots i_{n-1} \cdots j i_{n+1} \cdots i_N} = \sum_{i_n=1}^{I_n} x_{i_1 i_2 \cdots i_N} u_{j i_n}, \quad (2)$$

where $j = 1, \ldots, J_n;\ i_k = 1, \ldots, I_k;\ k = 1, \cdots, N.$

III. The Formulation of Tensor Completion

In this section, we introduce the low-rank and Tucker model for the tensor completion.

A. Low-Rank model for Tensor Completion

Before introducing the model of tensor completion problem, let's review the low-rank matrix completion problem. As we all known, the rank of the matrix is a general tool to capture the global information [20], and the model of low-rank matrix completion problem is as follows:

$$\begin{aligned} &\min_{\mathbf{X}} : \text{rank}(\mathbf{X}), \\ &s.t. : \mathbf{X}_\Omega = \mathbf{M}_\Omega, \end{aligned} \quad (3)$$

where the $\mathbf{X}, \mathbf{M} \in \mathbb{R}^{m \times n}$, Ω is the set of the indices of the observable elements of $\mathbf{M}$, and the remaining elements of $\mathbf{M}$ are missing. As the function rank$(\mathbf{X})$ is a nonconvex. S. Ma and D. Goldfarb have proposed that the rank$(\mathbf{X})$ in (3) can yields the nuclear norm minimization problem [21] as follows:

$$\begin{aligned} &\min_{\mathbf{X}} : ||\mathbf{X}||_* \\ &s.t. : \mathbf{X}_\Omega = \mathbf{M}_\Omega, \end{aligned} \quad (4)$$

where $||\cdot||_*$ is trace norm. As the tensor is the generalization of the matrix concept [7], so the tensor completion problem can be denoted as follows:

$$\begin{aligned} &\min_{\mathcal{X}} : ||\mathcal{X}||_* \\ &s.t. : \mathcal{X}_\Omega = \mathcal{M}_\Omega. \end{aligned} \quad (5)$$

Unlike matrix, computing the rank of a tensor is a NP hard problem [7] [22]. Therefore, problem (5) usually considered as a low-rank tensor completion problem model. Problem (5) is a convex optimization problem, but it should be solved by the singular value computation of the tensor, and its computational cost is very high, such as Tucker rank could be viewed as the extension of matrix rank, and for a k dimension tensor $\mathcal{X}$, we have to implement k times SVD operation, which is relatively computational cost, so it doesn't suitable for large-scale or higher-order data processing.

B. The Tucker model for Tensor Completion

In this subsection, we introduce a common heuristic model of tensor completion, it is Tucker model [23] for tensor factorization to the tensor completion [7] as follows:

$$\begin{aligned} &\min_{\mathcal{X}, \mathcal{C}, \mathbf{U}_1, \cdots, \mathbf{U}_n} : \frac{1}{2} ||\mathcal{X} - \mathcal{C} \times_1 \mathbf{U}_1 \times_2 \mathbf{U}_2 \times_3 \cdots \times_n \mathbf{U}_n||_F^2 \\ &s.t. :\ \mathcal{X}_\Omega = \mathcal{T}_\Omega, \end{aligned} \quad (6)$$

where $\mathcal{C} \times_1 \mathbf{U}_1 \times_2 \mathbf{U}_2 \times_3 \cdots \times_n \mathbf{U}_n$ is the Tucker model based tensor factorization, $\mathbf{U}_i \in \mathbb{R}^{I_i \times r_i}$, $\mathcal{C} \in \mathbb{R}^{r_1 \times \cdots \times r_n}$, and $\mathcal{T}, \mathcal{X} \in \mathbb{R}^{I_1 \times \cdots \times I_n}$.

Tucker model is a good choice for when one only concerns about the completion accuracy without caring about the u-niqueness or interpretability of the decomposed latent factors [24]. In this paper, we propose a novel orthogonal random projection method for tensor completion based on Tucker model, the details would be explained in the next section.

IV. Orthogonal Random Projection for Tensor Completion

A. The model of ORPTC

Based on (6), the tensor completion problem can be rewritten as follows:

$$\begin{aligned} &\min ||\mathcal{X}_\Omega - \mathcal{M}_\Omega||_F^2 \\ &s.t.\ \text{rank}(\mathbf{X}_{(n)}) \le R_{(n)}, \end{aligned} \quad (7)$$

where $\mathbf{X}_{(n)}$ is the n-mode unfolding of the $\mathcal{X}$, and $R_{(n)}$ is the multilinear rank of $\mathcal{X}$ [25].

$$\mathcal{X} = \mathcal{G} \times_1 \mathbf{A}_{(1)} \times_2 \mathbf{A}_{(2)} \times_3 \cdots \times_N \mathbf{A}_{(N)}, \quad (8)$$

where $\mathbf{A} = [\mathbf{A}_{(1)}, \mathbf{A}_{(2)}, \cdots, \mathbf{A}_{(N)}]$ are the factors of $\mathcal{X}$.

The problem (7) can be solved by the Alternating Least Squares (ALS) method [19], wherein

$$\mathcal{G} = \mathcal{X} \times_1 {\mathbf{A}_{(1)}}^T \times_2 {\mathbf{A}_{(2)}}^T \times_3 \cdots \times_N {\mathbf{A}_{(N)}}^T \quad (9)$$

Detailed optimization procedure is presented in Algorithm 1. Wherein m is the number of maximum iteration, and the ε is convergence threshold.

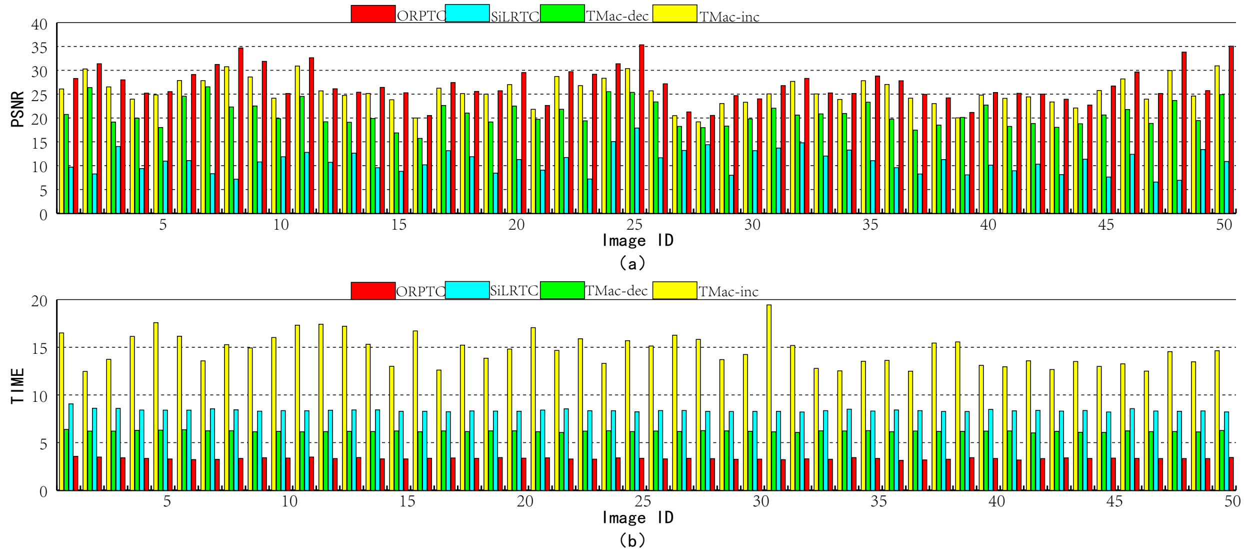

Fig. 2. Randomly selected 50 images with sampling rate of 0.7, after processing by TMac-inc, TMac-dec, SiLRTC and ORPTC, (a) comparison of the PSNR of the recovered image, (b) comparison of the running time of the experiment.

Algorithm 1 Alternating Least Squares for Tucker factorization approximation [19]

Input: $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$, m, ε
Output: $\mathbf{A}_{(i)}, i = 1, 2, \ldots, N$; $\mathcal{G} \in \mathbb{R}^{r_1 \times r_2 \times \cdots \times r_N}$
Initialize: randomize $\mathbf{A}_i, \mathcal{G}$

repeat
  For $n \in [1, m]$
    Update $\mathbf{A}_{(1)} \leftarrow \text{SVD}(\text{unfold}_1(\mathcal{X}))$
    Update $\mathbf{A}_{(2)} \leftarrow \text{SVD}(\text{unfold}_2(\mathcal{X}))$
    ......
    Update $\mathbf{A}_{(N)} \leftarrow \text{SVD}(\text{unfold}_N(\mathcal{X}))$
  end for
until $||\mathcal{X}^n - \mathcal{X}^{n-1}||_F^2 < \varepsilon$, or $n = m$
Update $\mathcal{G}$ by (8)
Update $\mathcal{X}$ by (7)

B. Orthogonal Random Projection for Tensor Completion

Solving problem (7) is usually time-consuming due to repeatedly computing the SVD of large unfolding matrices, which may limit its further application in large scale problems. In order to avoid the SVD operations, we propose a novel and fast randomized based algorithm called Orthogonal Random Projection for Tensor Completion (ORPTC). Before implementing proposed method, the unknown elements of tensor $\mathcal{M}$ are initialized with Gaussian distribution, this operation could be denoted by $\mathcal{X} = \text{full}(\mathcal{M})$.

In problem (7), as we need to compute the factor matrix $\mathbf{A}_{(i)}$, so we should initialize the n-mode unfolding matrix at first. The n-mode unfolding matrix is computed by

$$\mathbf{X}_n = \text{unfold}_n(\mathcal{X}). \tag{10}$$

In regular Tucker decomposition, to obtain the low rank approximation of n-mode unfolding matrix $\mathbf{X}_n$, QR decomposition would be incorporated. In this case, in order to obtain the left and right orthogonal basis matrix, we firstly generate the random projection matrix $\mathbf{H}_n \in \mathbb{R}^{I_1 I_2 \ldots I_{n-1} I_{n+1} \ldots I_N \times r_n}, r_n \ll I_n$, and obtain the low rank n-mode matrix as

$$\mathbf{C} = \mathbf{X}_n \mathbf{H}_n, \tag{11}$$

where $\mathbf{C}$ is the low rank randomized projection matrix. The left orthogonal basis matrix could be obtained by QR algorithm, e.g., Gram-Schmidt $\mathbf{A} = \text{QR}(\mathbf{C})$. Similarly, the right orthogonal basis matrix is solved by $\mathbf{V} = \text{QR}(\mathbf{X}_n^T \mathbf{A})$. And the core matrix could be computed as

$$\mathbf{M} = \mathbf{A}^T \mathbf{X}_n \mathbf{V}. \tag{12}$$

Therefore, the completed n-mode matrix $\mathbf{X}_n$ could be updated as

$$\mathbf{X}_n = \mathbf{A}\mathbf{M}\mathbf{V}^T. \tag{13}$$

The pseudocode of above ORPTC algorithm is presented in Algorithm 2. The convergence conditions are $||\mathcal{X}^n - \mathcal{X}^{n-1}||/||\mathcal{X}^n|| < \varepsilon$, and if the number of iterations reach to the maximum which we set.

V. Experiments

In this section, we compared the TMac-inc, TMac-dec, SiLRTC and ORPTC methods using images recovering experiments. In all experiments, we assume that the tensor which we need to complete is $\mathcal{M} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ with rank $r = (r_1, r_2, \cdots, r_N)$, In this paper, our initial ranks are [30,30,3] for the color images recovery. We set the stop

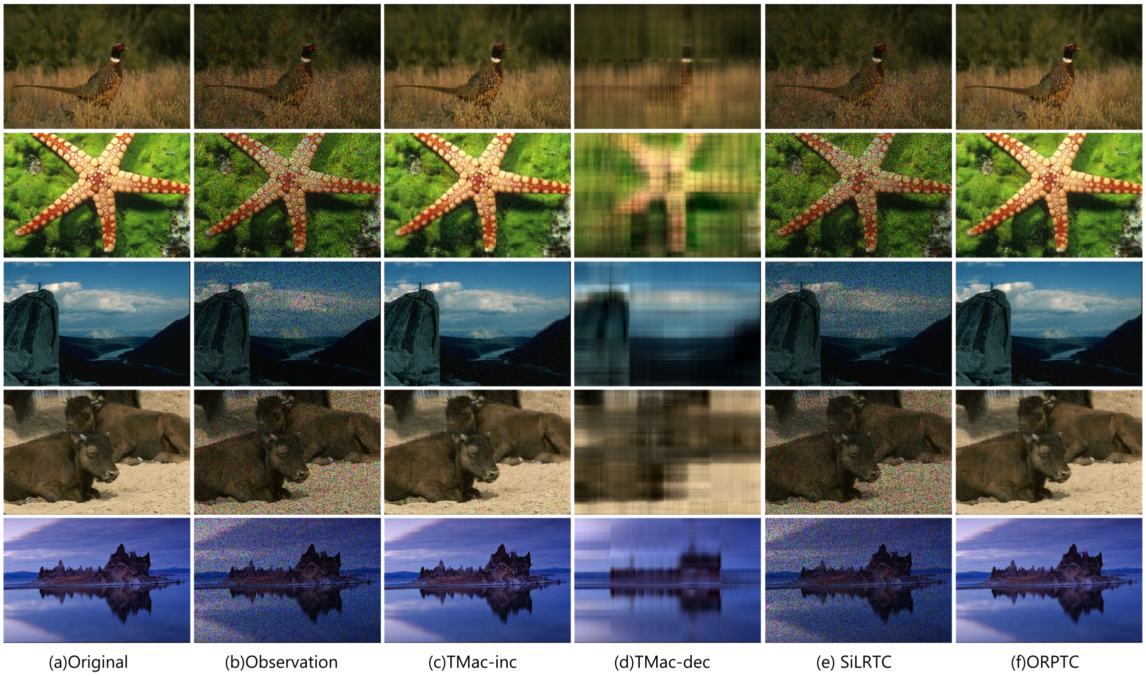

Fig. 3. (a) the original image, (b) image after sampling, and the sampling ratio is 0.7, (c) TMac-inc is mean Low-Rank Tensor Completion by Parallel Matrix Factorization with rank-increasing strategy, (d) TMac-dec is mean Low-Rank Tensor Completion by Parallel Matrix Factorization with rank-decreasing strategy, (e) SiLRTC is mean Simple low-Rank Tensor Completion, (f) ORPTC is mean Orthogonal Random Projection for Tensor Completion.

Algorithm 2 Orthogonal Random Projection for Tensor Completion (ORPTC)

Input: $\mathcal{M} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$; the observable elements' indices set Ω; m; ε

Output: $\mathcal{G}$; $\mathcal{X}$

Initialize: $\mathcal{X}^0 = \text{full}(\mathcal{M})$; $\mathbf{X}_{mode} = \text{unfold}_{mode}(\mathcal{X})$

repeat $n = 1, \cdots, m$ **do**
  for *mode*=1,...,N **do**
    $\mathbf{H} \in \mathbb{R}^{m \times n}$ $in\ i.i.d.\ N(0,1)$
    $\mathbf{C} = \mathbf{X}_{mode}\mathbf{H}$
    $\mathbf{A}_{(mode)} \leftarrow QR(\mathbf{C})$
    $\mathbf{R} = \mathbf{A}^T_{(mode)}\mathbf{X}_{mode}$
    $\mathbf{V}_{mode} \leftarrow QR(\mathbf{R}^T)$
    $\mathbf{M}_{mode} \leftarrow \mathbf{A}^T_{(mode)}\mathbf{X}_{mode}\mathbf{V}_{mode}$
    Update $\mathbf{X}_{mode} \leftarrow \mathbf{A}_{(mode)}\mathbf{M}_{mode}\mathbf{V}^T_{mode}$
  end for
until $||\mathbf{X}^n_{mode} - \mathbf{X}^{n-1}_{mode}||/||\mathbf{X}^n_{mode}|| < \varepsilon$ or reach maximum iterations exhausted conditions
$\mathcal{G}^n \leftarrow \mathcal{X}^{n-1} \times_1 \mathbf{A}^T_{(1)} \times_2 \mathbf{A}^T_{(2)} \times_3 \cdots \times_N \mathbf{A}^T_{(N)}$
$\mathcal{X}^n \leftarrow \mathcal{G}^n \times_1 \mathbf{A}_{(1)} \times_2 \mathbf{A}_{(2)} \times_3 \cdots \times_N \mathbf{A}_{(N)}$

condition as [6] for all the compared experiments methods as follows:

$$\frac{||V_1^k - V_1^{k-1}||}{||V_1^{k-1}||}, \frac{||V_2^k - V_2^{k-1}||}{||V_2^{k-1}||}, ..., \frac{||V_n^k - V_n^{k-1}||}{||V_n^{k-1}||} < \varepsilon \quad (14)$$

and if the iterative number reach to the maximum number of iterations. Here V_i is the i-th variable that we need to update, k is the number of iteration, and ε is the threshold of the stop criteria, we set $\varepsilon = 10^{-3}$ in this paper, and the maximum number of the iteration is 150. The experiments were carried out in MATLAB 2016b, on a server equipped with two E5-264014, 128G of RAM.

A. Real Data Experiments

We evaluate our method with color images recovery, as the color image has three channels, red, green, and blue channels, an image forms as a 3-way tensor. In order to evaluate the image recovery quality of TMac-inc, TMac-dec, SiLRTC and ORPTC method, we employ the peak signal-to-noise ratio (PSNR) [6] defined as

$$PSNR = 10log_{10}(\frac{I_1 I_2 I_3 ||\mathcal{M}||^2_\infty}{||\hat{\mathcal{X}} - \mathcal{M}||^2_F}), \quad (15)$$

where $\hat{\mathcal{X}}$ is the recovered tensor of $\mathcal{M} \in \mathbb{R}^{I_1 \times I_2 \times I_3}$, $||\mathcal{M}||_\infty$ means the absolute value maximum of the $\mathcal{M}$, $||\mathcal{X}||_F$ is the Frobenius norm of $\mathcal{X}$. Besides PSNR, the computational time is also very important criterion, so we also use the running time to compare their computational cost.

In this experiment, all the color images are from Berkeley Segmentation database [26]. There are 200 color images in the database, each size is 321 × 481 × 3. We select 50 color images from those database randomly, and set the sampling ratio p = 0.7, which means 70% pixel values are known, and

30% pixel values are set to zero. As [6], we set the initial rank of TMac with rank-decreasing strategy are [30,30,30], and the initialized rank of TMac with rank-increasing strategy are [3,3,3], and the parameter $\alpha = [1, 1, 1]$ and $\beta = [1, 1, 1]$ for SiLRTC. In Fig. 2, we report the PSNR values and running time for the 50 color images which after testing by those methods.

TABLE I
PSNR AND RUNNING TIME (SECONDS) USING THE ABOVE FIVE IMAGES

Image	TMac-inc		TMac-dec		SiLRTC		ORPTC	
	PSNR	time	PSNR	time	PSNR	time	PSNR	time
1	28.07	17.75	25.53	6.02	17.49	8.68	**32.77**	**4.94**
2	24.74	18.75	18.01	6.07	12.35	8.55	**27.23**	**4.59**
3	31.05	15.30	24.54	6.03	14.59	8.79	**34.00**	**4.72**
4	27.44	12.58	22.49	6.08	12.80	8.76	**31.19**	**4.62**
5	30.44	19.59	24.90	5.99	12.41	8.90	**35.45**	**4.59**

We show the inpainting result of five of images tested in Fig. 3 and table 1. We can see that ORPTC performs the best. Our method not only is the fastest , but also has the highest PSNR in those experiments.

VI. CONCLUSIONS

In this paper, we employ the model of Tucker decomposition to reconstruct the tensor which is including unknown elements. We use the orthogonal random projection method instead of singular value decomposition to compute the factor matrices. ORPTC greatly cut down the time that required for tensor completion, hence it is very suitable for large-scale problems.

REFERENCES

[1] B. Recht, M. Fazel, and P. A. Parrilo, "Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization," *SIAM review*, vol. 52, no. 3, pp. 471–501, 2010.

[2] E. J. Candès and B. Recht, "Exact matrix completion via convex optimization," *Foundations of Computational mathematics*, vol. 9, no. 6, p. 717, 2009.

[3] J.-F. Cai, E. J. Candès, and Z. Shen, "A singular value thresholding algorithm for matrix completion," *SIAM Journal on Optimization*, vol. 20, no. 4, pp. 1956–1982, 2010.

[4] B. Recht, "A simpler approach to matrix completion," *Journal of Machine Learning Research*, vol. 12, no. Dec, pp. 3413–3430, 2011.

[5] E. J. Candès and T. Tao, "The power of convex relaxation: Near-optimal matrix completion," *IEEE Transactions on Information Theory*, vol. 56, no. 5, pp. 2053–2080, 2010.

[6] P. Zhou, C. Lu, Z. Lin, and C. Zhang, "Tensor factorization for low-rank tensor completion," *IEEE Transactions on Image Processing*, vol. 27, no. 3, pp. 1152–1163, 2018.

[7] J. Liu, P. Musialski, P. Wonka, and J. Ye, "Tensor completion for estimating missing values in visual data," *IEEE transactions on pattern analysis and machine intelligence*, vol. 35, no. 1, pp. 208–220, 2013.

[8] C. Lu, J. Feng, Y. Chen, W. Liu, Z. Lin, and S. Yan, "Tensor robust principal component analysis: Exact recovery of corrupted low-rank tensors via convex optimization," in *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*, 2016, pp. 5249–5257.

[9] S. Gandy, B. Recht, and I. Yamada, "Tensor completion and low-n-rank tensor recovery via convex optimization," *Inverse Problems*, vol. 27, no. 2, p. 025010, 2011.

[10] M. Mørup, "Applications of tensor (multiway array) factorizations and decompositions in data mining," *Wiley Interdisciplinary Reviews: Data Mining and Knowledge Discovery*, vol. 1, no. 1, pp. 24–40, 2011.

[11] A. Karatzoglou, X. Amatriain, L. Baltrunas, and N. Oliver, "Multiverse recommendation: n-dimensional tensor factorization for context-aware collaborative filtering," in *Proceedings of the fourth ACM conference on Recommender systems*. ACM, 2010, pp. 79–86.

[12] M. Bertalmio, G. Sapiro, V. Caselles, and C. Ballester, "Image inpainting," in *Proceedings of the 27th annual conference on Computer graphics and interactive techniques*. ACM Press/Addison-Wesley Publishing Co., 2000, pp. 417–424.

[13] C. Lu, J. Feng, Y. Chen, W. Liu, Z. Lin, and S. Yan, "Tensor robust principal component analysis with a new tensor nuclear norm," *arXiv preprint arXiv:1804.03728*, 2018.

[14] Z. Zhang, G. Ely, S. Aeron, N. Hao, and M. Kilmer, "Novel methods for multilinear data completion and de-noising based on tensor-svd," in *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*, 2014, pp. 3842–3849.

[15] M. E. Kilmer, K. Braman, N. Hao, and R. C. Hoover, "Third-order tensors as operators on matrices: A theoretical and computational framework with applications in imaging," *SIAM Journal on Matrix Analysis and Applications*, vol. 34, no. 1, pp. 148–172, 2013.

[16] T.-H. Oh, Y. Matsushita, Y.-W. Tai, and I. S. Kweon, "Fast randomized singular value thresholding for low-rank optimization," *IEEE transactions on pattern analysis and machine intelligence*, vol. 40, no. 2, pp. 376–391, 2018.

[17] Y. Wang, H.-Y. Tung, A. J. Smola, and A. Anandkumar, "Fast and guaranteed tensor decomposition via sketching," in *Advances in Neural Information Processing Systems*, 2015, pp. 991–999.

[18] C. E. Tsourakakis, "Mach: Fast randomized tensor decompositions," in *Proceedings of the 2010 SIAM International Conference on Data Mining*. SIAM, 2010, pp. 689–700.

[19] T. G. Kolda and B. W. Bader, "Tensor decompositions and applications," *SIAM review*, vol. 51, no. 3, pp. 455–500, 2009.

[20] H. Tan, B. Cheng, W. Wang, Y.-J. Zhang, and B. Ran, "Tensor completion via a multi-linear low-n-rank factorization model," *Neurocomputing*, vol. 133, pp. 161–169, 2014.

[21] S. Ma, D. Goldfarb, and L. Chen, "Fixed point and bregman iterative methods for matrix rank minimization," *Mathematical Programming*, vol. 128, no. 1-2, pp. 321–353, 2011.

[22] C. J. Hillar and L.-H. Lim, "Most tensor problems are np-hard," *Journal of the ACM (JACM)*, vol. 60, no. 6, p. 45, 2013.

[23] L. R. Tucker, "Some mathematical notes on three-mode factor analysis," *Psychometrika*, vol. 31, no. 3, pp. 279–311, 1966.

[24] Q. Song, H. Ge, J. Caverlee, and X. Hu, "Tensor completion algorithms in big data analytics," *arXiv preprint arXiv:1711.10105*, 2017.

[25] M. Che and Y. Wei, "Randomized algorithms for the approximations of tucker and the tensor train decompositions," *Advances in Computational Mathematics*, pp. 1–34, 2018.

[26] D. Martin, C. Fowlkes, D. Tal, and J. Malik, "A database of human segmented natural images and its application to evaluating segmentation algorithms and measuring ecological statistics," in *Computer Vision, 2001. ICCV 2001. Proceedings. Eighth IEEE International Conference on*, vol. 2. IEEE, 2001, pp. 416–423.