Random Signal Estimation by Ergodicity associated with Linear Canonical Transform

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***Abstract*—The linear canonical transform (LCT) provides a general mathematical tool for solving problems in optical and quantum mechanics. For random signals, which are bandlimited in the LCT domain, the linear canonical correlation function and the linear canonical power spectral density can form a LCT pair. The linear canonical translation operator, which is used to define the convolution and correlation functions, also plays a significant role in the analysis of the random signal estimation. Firstly, the eigenfunctions which are invariant under the linear canonical translation and the unitarity property of it are discussed. Secondly, it shows that all of these connect the LCT sampling theorem and the von Neumann ergodic theorem in the sense of distribution, which will develop an estimation method for the power spectral density of a chirp stationary random signal from one sampling signal in the LCT domain. Finally, the potential applications and future work are discussed.**

***Index Terms*—Linear canonical transform, Random signal, Power spectral density, Ergodicity, Quantum mechanics**

I. Introduction

The linear canonical transform (LCT) provides a general mathematical tool for solving problems in nonstationary signal processing, radar, sonar, optical and quantum mechanics [1-4]. It is a four parameter (a, b, c, d) class of linear integral transform and many well-known signal processing operations are its special cases, such as the Fourier transform (FT), fractional Fourier transform (FRFT), Fresnel transform and scaling operations. The deterministic signal analysis in the LCT domain has had a researchful study, such as the convolution theorem, the correlation function, the sampling theorem and so on [5-10]. The random signal analysis in the LCT domain has also been researched in several literatures [11-16].

In the stochastic signal processing, a stochastic process is said to be stationary in a wide sense when it has a zero mean and its auto-correlation function does not change over time [11-13]. The Wiener-Khinchine theorem is an important theorem in the random signal analysis, which states that the power spectrum density and the correlation function form a FT pair. Similarly, it has been proved that for a nonstationary signal, if it is chirp stationary in a LCT domain, the linear canonical correlation function and the linear canonical power spectral density of the signal form a LCT pair [13].

As is known to all, the convolution and correlation functions for a transform operation can be defined using the time-shift operator. Jun Shi et al. have proposed the generalized convolution and product theorems associated with LCT by introducing a linear canonical translation operator [5]. Actually, the translation operator defined in [5] includes the shifts in time, frequency and linear modulated frequency. The ergodic theory has been introduced by Boltzmann in the context of statistical mechanics, which is the study of the conditions that permit to change a temporal average by an ensemble average [17]. The von Neumann's theorem considered advantages in spectral resolution of linear operators, such as the FT and consequently in the harmonic analysis. Sampling theorem plays an important role in the digital signal processing. The sampling theorems and error estimates for random signals in the linear canonical transform domain have been studied in the mean square sense [13]. In this paper, we will first discuss the properties of the linear canonical translation operators which has been defined in [5] in detail, then establish the relationship between the sampling theorem and the von Neumann ergodic theorem, and last estimate the power spectral density of a random signal in the LCT domain in the sense of distributions.

The paper is organized as follows: Section II reviews the preliminaries about the LCT and the nonstationary random signal analysis in the LCT domain. In Section III, the properties of the linear canonical translation operator are researched first. Then, the von Neumann ergodic theorem for linear canonical translation and random signal estimation by ergodicity are proposed. Section IV concludes the paper.

II. Preliminaries

A. The linear canonical transform

The LCT with real parameter $\mathbf{M} = (a, b, c, d)$ of a signal $f(t)$ is defined as [1]

$$\begin{aligned} F_{\mathbf{M}}(u) &= L_{\mathbf{M}}[f(t)](u) \\ &= \begin{cases} B_{\mathbf{M}} \int_{-\infty}^{\infty} f(t) K_{\mathbf{M}}(u,t) dt, & b \neq 0 \\ \sqrt{d} e^{j(cd/2\,)u^2} f(du), & b = 0 \end{cases} \end{aligned} \tag{1}$$

where $K_{\mathbf{M}}(u,t) = e^{j\frac{1}{2}(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2)}$ is the transform kernel, $B_{\mathbf{M}} = \sqrt{1/(j2\pi b)}$, and $\det(\mathbf{M}) = ad - bc = 1$. The inverse transform of the LCT is given by the LCT with parameter $\mathbf{M}^{-1} = (d, -b, -c, a)$. From the definition of the LCT, it is obvious that the LCT with special parameters could reduce to the transforms what are familiar, such as the FT when $\mathbf{M} = (0, 1, -1, 0)$, the FRFT when $\mathbf{M} = (\cos\alpha, \sin\alpha, -\sin\alpha, \cos\alpha)$, the Fresnel transform when $\mathbf{M} =$

$(1, b, 0, 1)$ and so on. More details about the properties of the LCT can be found in the references [1-5].

Using the quantum mechanical notation [4], the LCT of the signal $f(t)$ can be written as

$$\begin{aligned} L_{\mathbf{M}}[f(t)](u) &= B_{\mathbf{M}} \int_{-\infty}^{\infty} dt \, \langle u| K_{\mathbf{M}} |t\rangle \langle t|f\rangle \\ &= B_{\mathbf{M}} \langle u| K_{\mathbf{M}} |f\rangle \\ &= \langle u|F_{\mathbf{M}}\rangle \\ &= F_{\mathbf{M}}(u), \end{aligned} \tag{2}$$

where the function is denoted as $f(t) = \langle t|f\rangle$ and $\langle u| K_{\mathbf{M}} |t\rangle = K_{\mathbf{M}}(u, t)$ gives the representation of the LCT kernel in quantum mechanics.

B. Nonstationary random signal analysis in LCT domain

The stationarity is one of the most important concepts in random signal processing. For a nonstationary signal $x(t)$, if its modulated form $x(t)e^{jat^2/2b}$ is stationary in the wide sense, the signal is said to be chirp stationary [11-13]. The LCT auto-correlation function is defined as [13]

$$R_{xx}^{\mathbf{M}}(t_1, t_2) = E\{x(t_1)x^*(t_2)e^{jat_2(t_1-t_2)/b}\}. \tag{3}$$

When the random signal is linear canonical chirp stationary, we have that

$$R_{xx}^{\mathbf{M}}(t_1, t_2) = R_{xx}^{\mathbf{M}}(\tau)\,|_{\tau=t_1-t_2}\,. \tag{4}$$

The LCT power spectral density and the LCT auto-correlation function form a LCT pair as follows

$$P_{xx}^{\mathbf{M}}(u) = \sqrt{\frac{1}{-j2\pi b}} L^{\mathbf{M}}[R_{xx}^{\mathbf{M}}(\tau)](u)e^{-jdu^2/(2b)}\,. \tag{5}$$

Obviously, the result reduces to the Wiener-Khinchine theorem when $\mathbf{M} = (0, 1, -1, 0)$.

III. MAIN RESULTS

A. The linear canonical translation operator

The time-shift operator plays a significant role in the definition of convolution and correlation functions [5]. In this subsection, we introduce the linear canonical translation operator and study its properties first, which will be used in the future work.

The linear canonical translation operator $T_{\tau,\mathbf{M}}$ with parameter $\mathbf{M} = (a, b, c, d)$ and value τ is defined as [5]

$$T_{\tau,\mathbf{M}}[f](t) = f(t-\tau)e^{-j\tau(t-\tau/2\,)a/b}\,. \tag{6}$$

It satisfies the following property based on the definition of the LCT

$$L_{\mathbf{M}}\{T_{\tau,\mathbf{M}}[f]\}(u) = F_{\mathbf{M}}(u)e^{-ju\tau/b}\,, \tag{7}$$

where $F_{\mathbf{M}}(u)$ is the LCT of $f(t)$. When $\mathbf{M} = (0, 1, -1, 0)$, the operator $T_{\tau,\mathbf{M}}$ reduces to the ordinary time-shift operator $T_\tau[f](t) = f(t - \tau)$.

It is easy to prove that the linear canonical translation operator forms a commutative group which holds the following group property

$$T_{\tau,\mathbf{M}} \circ T_{\upsilon,\mathbf{M}} = T_{\tau+\upsilon,\mathbf{M}}. \tag{8}$$

Therefore,

$$T_{n\tau,\mathbf{M}} = T_{\tau,\mathbf{M}} \circ T_{\tau,\mathbf{M}} \circ \cdots \circ T_{\tau,\mathbf{M}} = T^n_{\tau,\mathbf{M}}. \tag{9}$$

As a result, the group of the linear canonical translations $\{T_{\tau,\mathbf{M}} : -\infty < \tau < \infty\}$ is a real-parameter group, and its inverse operator is $T^{-1}_{\tau,\mathbf{M}} = T_{-\tau,\mathbf{M}}$ and identity operator is $T_{0,\mathbf{M}} = I$.

Next, we analyze the properties of the linear canonical translation.

(a) *The eigenfunctions of linear canonical translation*

For the set of the orthonormal functions

$$\varphi_{x,\mathbf{M}}(x') = B_{\mathbf{M}} e^{j\frac{1}{2}(\frac{a}{b}x^2 - \frac{2}{b}xx' + \frac{d}{b}x'^2)}, \tag{10}$$

it can be shown that these functions satisfy the following eigenvalue equation:

$$\begin{aligned} & T_{\tau,\mathbf{M}^{-1}}[\varphi_{x,\mathbf{M}}](x') \\ = \;& \varphi_{x,\mathbf{M}}(x'-\tau)e^{j\tau(x'-\tau/2\,)d/b} \\ = \;& B_{\mathbf{M}} e^{j\frac{1}{2}(\frac{a}{b}x^2 - \frac{2}{b}x(x'-\tau) + \frac{d}{b}(x'-\tau)^2)} e^{j\tau(x'-\frac{\tau}{2})\frac{d}{b}} \\ = \;& B_{\mathbf{M}} e^{j\frac{1}{2}(\frac{a}{b}x^2 - \frac{2}{b}xx' + \frac{2}{b}x\tau + \frac{d}{b}x'^2)} \\ = \;& e^{j\frac{1}{b}x\tau}\varphi_{x,\mathbf{M}}(x'). \end{aligned} \tag{11}$$

Therefore, the functions $\varphi_{x,\mathbf{M}}(x')$ are the eigenfunctions of the operator $T_{\tau,\mathbf{M}^{-1}}$ with the eigenvalue $\lambda_{x,\mathbf{M}} = e^{j\frac{1}{b}x\tau}$.

(b) *The invariant subspace of linear canonical translation*

The following functions are the subset of the eigenfunctions of $T_{\tau,\mathbf{M}^{-1}}$,

$$\varphi_{2\pi n\frac{b}{\tau},\mathbf{M}}(x') = B_{\mathbf{M}} e^{j\frac{1}{2}((2\pi n)^2\frac{ab}{\tau^2} - 2\pi n\frac{2}{\tau}x' + \frac{d}{b}x'^2)}, \tag{12}$$

where n is an arbitrary integer. The corresponding eigenvalue is $\lambda_{2\pi n\frac{b}{\tau},\mathbf{M}} = 1$, i.e.

$$T_{\tau,\mathbf{M}^{-1}}[\varphi_{2\pi n\frac{b}{\tau},\mathbf{M}}](x') = \varphi_{2\pi n\frac{b}{\tau},\mathbf{M}}(x'). \tag{13}$$

These functions are invariant functions under the linear canonical translation $T_{\tau,\mathbf{M}^{-1}}$. It can be proved that they have the property of orthogonality. For the functions $\varphi_{2\pi n\frac{b}{\tau},\mathbf{M}}$ and $\varphi_{2\pi m\frac{b}{\tau},\mathbf{M}}$, we calculate

$$\begin{aligned} & \int_{-\tau/2}^{\tau/2} \varphi_{2\pi n\frac{b}{\tau},\mathbf{M}}(x)\varphi^*_{2\pi m\frac{b}{\tau},\mathbf{M}}(x)dx \\ = \;& \frac{1}{2\pi b} e^{j\frac{1}{2}[(2\pi n)^2 - (2\pi m)^2]\frac{ab}{\tau^2}} \int_{-\tau/2}^{\tau/2} e^{-j(2\pi n - 2\pi m)\frac{1}{\tau}x}dx. \end{aligned} \tag{14}$$

And because of

$$\int_{-\tau/2}^{\tau/2} e^{-j[2\pi n - 2\pi m]\frac{1}{\tau}x}dx = \tau \cdot \delta_{n,m}, \tag{15}$$

we can obtain

$$\begin{aligned}&\int_{-\tau/2}^{\tau/2}\varphi_{2\pi n\frac{b}{\tau},\mathbf{M}}(x)\varphi^*_{2\pi m\frac{b}{\tau},\mathbf{M}}(x)dx\\ =&\ \frac{\tau}{2\pi b}e^{j\frac{1}{2}[(2\pi n)^2-(2\pi m)^2]\frac{ab}{\tau^2}}\delta_{n,m}.\end{aligned}\tag{16}$$

That is to say, the invariant functions $\varphi_{2\pi n\frac{b}{\tau},\mathbf{M}}(x)$ under the linear canonical translation $T_{\tau,\mathbf{M}^{-1}}$ are orthonormal, and constitute a complete set in $x\in[-\tau/2\ ,\tau/2\ ]$.

(c) *The unitarity of linear canonical translation*

The linear canonical translation can be defined by the integral form

$$T_{\tau,\mathbf{M}}[f](x)=\int_R f(u)K_{\tau,\mathbf{M}}(x,u)du,\tag{17}$$

where the kernel is

$$K_{\tau,\mathbf{M}}(x,u)=\delta(x-\tau-u)e^{-j\frac{a}{b}u(x-u)}e^{-j\frac{a}{2b}\tau^2}.\tag{18}$$

The evaluation of the Hermitian adjoint of $K_{\tau,\mathbf{M}}(x,u)$ leads to

$$\begin{aligned}K^\dagger_{\tau,\mathbf{M}}(x,u)=&\ \overline{K_{\tau,\mathbf{M}}(u,x)}\\ =&\ \delta(u-\tau-x)e^{j\frac{a}{b}x(u-x)}e^{j\frac{a}{2b}\tau^2},\end{aligned}\tag{19}$$

from where we obtain

$$\int_R f(u)K^\dagger_{\tau,\mathbf{M}}(x,u)du=T_{-\tau,\mathbf{M}}[f](x).\tag{20}$$

Consequently, the linear canonical translation is an unitary operator and satisfies the identity

$$T^\dagger_{\tau,\mathbf{M}}\circ T^{-1}_{\tau,\mathbf{M}}=T^{-1}_{\tau,\mathbf{M}}\circ T^\dagger_{\tau,\mathbf{M}}=I.\tag{21}$$

B. The von Neumann ergodic theorem for linear canonical translation

The ergodic theorem due to von Neumann states that for a given unitary operator U on the Hilbert space $H=L_2(\Omega,F,\mathbf{P})$, where $(\Omega,F,\mathbf{P})$ is a probability space, in general U can be any isometric operator on H, and the orthogonal projection $\mathbf{P}$ onto the subspace of all function invariant under U, $\{\varphi\in H\,|U\varphi=\varphi\}$; the following limit

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}U^nf=\mathbf{P}f=E\{f\,|F_\mathbf{I}\}\tag{22}$$

exists for any $f\in H$ in the sense of the norm convergence in H. In other terms, $\frac{1}{N}\sum\limits_{n=0}^{N-1}U^n$ converges to $\mathbf{P}$ in the strong operator topology [17].

For a wide sense stationary random signal $U(x)$, which is bandlimited with band width $u_\mathbf{M}$ in the LCT domain with parameter $\mathbf{M}$, the linear canonical auto-correlation function $R^\mathbf{M}$ is also a random function. Considering the sampled version of $R^\mathbf{M}$ as follows

$$\tilde{R}^\mathbf{M}(x)=\frac{2\pi b}{u_\mathbf{M}}\sum_{n=-\infty}^{\infty}R^\mathbf{M}(n\frac{2\pi b}{u_\mathbf{M}})\delta(x-n\frac{2\pi b}{u_\mathbf{M}}),\tag{23}$$

where the sampling interval is $\frac{2\pi b}{u_\mathbf{M}}$. Then,

$$\begin{aligned}&L_\mathbf{M}[\tilde{R}^\mathbf{M}](u)\\ =&\ \frac{2\pi bB_\mathbf{M}}{u_\mathbf{M}}\sum_{n=-\infty}^{\infty}R^\mathbf{M}(n\frac{2\pi b}{u_\mathbf{M}})e^{j\frac{1}{2}[(2\pi n)^2\frac{ab}{u^2_\mathbf{M}}-2\pi n\frac{2u}{u_\mathbf{M}}+\frac{d}{b}u^2]}.\end{aligned}\tag{24}$$

On the other hand, if we use the Poisson summation formula

$$\sum_n\delta(t-nT)=\sum_k\frac{1}{T}e^{jk\frac{2\pi}{T}t},\tag{25}$$

(23) can be rewritten as

$$\tilde{R}^\mathbf{M}(x)=R^\mathbf{M}(x)\sum_{n=-\infty}^{\infty}e^{jnu_\mathbf{M}x/b}.\tag{26}$$

Then, in combination with the frequency shift property of LCT

$$\begin{aligned}&L_\mathbf{M}[e^{j\mu x}f(x)](u)\\ =&\ L_\mathbf{M}[f(x)](u-b\mu)e^{jd\mu u}e^{-jbd\mu^2/2},\end{aligned}\tag{27}$$

and the linear canonical translation (6), we can obtain

$$\begin{aligned}&L_\mathbf{M}[\tilde{R}^\mathbf{M}](u)\\ =&\ \sum_{n=-\infty}^{\infty}L_\mathbf{M}[\tilde{R}^\mathbf{M}](u-nu_\mathbf{M})e^{j\frac{d}{b}(nu_\mathbf{M}u)}e^{-j\frac{d}{2b}(nu_\mathbf{M})^2}\\ =&\ \sum_{n=-\infty}^{\infty}T_{nu_\mathbf{M},\mathbf{M}^{-1}}L_\mathbf{M}[\tilde{R}^\mathbf{M}](u).\end{aligned}\tag{28}$$

From the result of (24) and (28), by using the property of linear canonical translation in (9), we obtain

$$\begin{aligned}&\sum_{n=-\infty}^{\infty}T_{nu_\mathbf{M},\mathbf{M}^{-1}}L_\mathbf{M}[\tilde{R}^\mathbf{M}](u)\\ =&\ \sum_{n=-\infty}^{\infty}T^n_{u_\mathbf{M},\mathbf{M}^{-1}}L_\mathbf{M}[\tilde{R}^\mathbf{M}](u)\\ =&\ B_\mathbf{M}\frac{2\pi b}{u_\mathbf{M}}\sum_{n=-\infty}^{\infty}R^\mathbf{M}(n\frac{2\pi b}{u_\mathbf{M}})e^{j\frac{1}{2}[(2\pi n)^2\frac{ab}{u^2_\mathbf{M}}-2\pi n\frac{2u}{u_\mathbf{M}}+\frac{d}{b}u^2]}.\end{aligned}\tag{29}$$

We can see that the above expression closely resembles the form of the von Neumann Mean Ergodic theorem in (22).

C. Random signal estimation by ergodicity

Utilizing the representations in quantum mechanics, denote a function in the position representation as $\varphi_{2\pi nb/\tau\ ,\mathbf{M}}(x')=\langle x'|\varphi_{2\pi nb/\tau\ ,\mathbf{M}}\rangle$, then we can write

$$\left|L_\mathbf{M}R^\mathbf{M}\right\rangle=\int_R R^\mathbf{M}(x)\left|\varphi_{2\pi nb/u_\mathbf{M}\ ,\mathbf{M}}\right\rangle dx.\tag{30}$$

The above expression is in general true for random signals whose auto-correlation function $R^\mathbf{M}(x)\in H$, where $H=L_2(\Omega,F,P)$ is a Hilbert space. Nevertheless, the above identity is fulfilled in the sense of distribution.

The orthogonal projection operator onto the space of all vectors invariant under $T_{u_{\mathbf{M}},\mathbf{M}^{-1}}$ is given by

$$\mathbf{P}_{u_{\mathbf{M}},\mathbf{M}} = \sum_{n=-\infty}^{\infty} \left|\varphi_{2\pi nb/u_{\mathbf{M}}},\mathbf{M}\right\rangle \left\langle \varphi_{2\pi nb/u_{\mathbf{M}}},\mathbf{M}\right|, \tag{31}$$

and considering the following result

$$\left\langle \varphi_{2\pi nb/u_{\mathbf{M}}},\mathbf{M} \middle| \varphi_{x,\mathbf{M}} \right\rangle = \delta(2\pi nb/u_{\mathbf{M}} - x), \tag{32}$$

we have

$$\begin{aligned} & B_{\mathbf{M}} \frac{2\pi b}{u_{\mathbf{M}}} \sum_{n=-\infty}^{\infty} R^{\mathbf{M}}(n\frac{2\pi b}{u_{\mathbf{M}}}) e^{j\frac{1}{2}[(2\pi n)^2 \frac{ab}{u_{\mathbf{M}}^2} - 2\pi n \frac{2u}{u_{\mathbf{M}}} + \frac{d}{b}u^2]} \\ & = \frac{2\pi b}{u_{\mathbf{M}}} \left\langle u \right| \mathbf{P}_{u_{\mathbf{M}},\mathbf{M}} \left| L_{\mathbf{M}} R^{\mathbf{M}} \right\rangle. \end{aligned} \tag{33}$$

Substituting (33) into (29), we can obtain that

$$\sum_{n=-\infty}^{\infty} T^n_{u_{\mathbf{M}},\mathbf{M}^{-1}} \left| L_{\mathbf{M}} R^{\mathbf{M}} \right\rangle = \frac{2\pi b}{u_{\mathbf{M}}} \mathbf{P}_{u_{\mathbf{M}},\mathbf{M}} \left| L_{\mathbf{M}} R^{\mathbf{M}} \right\rangle. \tag{34}$$

The result is analogous to the von Neumann ergodic theorem of (22), but is in the sense of distribution. As the linear canonical correlation and LCT power spectral density form a LCT pair, the result of (34) can be rewritten as

$$\sum_{n=-\infty}^{\infty} T^n_{u_{\mathbf{M}},\mathbf{M}^{-1}} \left| P^{\mathbf{M}} \right\rangle = \frac{2\pi b}{u_{\mathbf{M}}} \mathbf{P}_{u_{\mathbf{M}},\mathbf{M}} \left| P^{\mathbf{M}} \right\rangle. \tag{35}$$

Therefore, the power spectral density of a random signal can be estimated by the ergodicity from the power spectral density of the sampling signal in the LCT domain.

D. Potential applications and future work

The estimation method of the power spectral density of a random signal from its sampling signal has been discussed. In practical applications, the signals are usually analyzed with random property as the influence by the external factors, such as in the applications of sea clutter suppression and micromotion marine target detection [18]. Some detection and estimation methods based on LCT has been proposed and verified to be effective in the detection of marine target with micromotion [19]. The method discussed in this paper may be also effective in some similar applications, which will be our future work.

IV. Conclusion

In the paper, the linear canonical translation operator which can be used to define generalized convolution and product theorems in the LCT domain is analyzed first, including its eigenfunctions, the invariant subspace and unitarity. Associating the von Neumann ergodic theorem with the linear canonical translation operator, the power spectral estimation of the random signal in the LCT domain is proposed by the ergodicity in the distribution sense.

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