Cohen's class time-frequency representation in linear canonical domains: definition and properties

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***Abstract*—The traditional Cohen's class time-frequency representation is extended to the linear canonical domain by using a well-established closed-form instantaneous cross-correlation function (CICF) type of linear canonical transform (LCT) free parameters embedded approach. The derived CICF type of Cohen's class (CICFCC) unifies some well-known Cohen's classes in linear canonical domains including the affine characteristic, basis function, convolution expression and instantaneous cross-correlation function types of Cohen's classes, and can be considered as the Cohen's class's closed-form representation in linear canonical domains. A fundamental theory about the CICFCC's essential properties, such as marginal distribution, energy conservation, unique reconstruction, Moyal formula, complex conjugate symmetry, time reversal symmetry, scaling property, time shift property, frequency shift property, and LCT invariance, is then established. Possible applications are also carried out to illustrate that the CICFCC outperforms the traditional one in non-stationary signal separation and detection.**

I. Introduction

Linear canonical transform (LCT) [1], also known as ABCD transform, general Fresnel transform, Collins formula, general Huygens integral, and Moshinsky and Quesne integrals [2], has three free parameters so that it offers more degrees of freedom than its special cases including the Fourier transform (FT) [3] and the fractional Fourier transform (FRFT) [4], [5]. From the view of signal representation, the LCT provides a kind of time-frequency domain, namely, the linear canonical domains that are flexible enough to analyse non-stationary signals [6].

Cohen's class time-frequency representation [7], a celebrated bilinear time-frequency distribution class, includes particular cases some well-known quadratic time-frequency distributions, such as Wigner distribution (WD) [8], Kirkwood-Rihaczek distribution (KRD) [9], Born-Jordan distribution (BJD) [10], Zhao-Atlas-Marks distribution (ZAMD) [11], and Choi-Williams distribution (CWD) [12]. It turns out that these time-frequency analysis tools are suitable for the process of non-stationary single signals including linear frequency-modulated (LFM) signals [13], quadratic frequency-modulated (QFM) signals [14], and sinusoidal frequency-modulated (SFM) signals [15]. As for non-stationary deterministic signals mixed lots of components or random signals corrupted heavily with noises, however, it becomes lack of flexibility for them to deal with. To meet the requirement of signal representation's flexibility for multi-component or strong noise interference in non-stationary signal processing, an effective technology method might be to introduce LCT free parameters into the Cohen's class.

There are many kinds of LCT free parameters embedded approaches that are proposed to generalize the WD into linear canonical domains from different perspectives, including affine characteristic (AC) [16], [17], basis function (BF) [18]–[20], convolution expression (CE) [21], instantaneous cross-correlation function (ICF) [22] and closed-form instantaneous cross-correlation function (CICF) [23], [24] types of methods. It is shown that the CICF type of parameters embedded approach unifies all of them, and can be regarded as a kind of closed-form method [23], [24]. It is therefore the main LCT free parameters embedded approach what we focus on in this work. One of the main contributions of this paper is thus to define the closed-form Cohen's class in linear canonical domains namely as the CICF type of Cohen's class (CICFCC) through the well-established CICF type of LCT free parameters embedded approach. It then obtains many useful and important properties of the CICFCC. It also studies the CICFCC's applications in non-stationary signal separation and detection.

The rest of this paper is organized as follows. Section II recalls some background and notation on the LCT briefly. In Section III, the definition of CICFCC is proposed. Its essential properties are also discussed in this section. Potential applications are performed in Section IV. Finally, Section V draws a conclusion, and collects some future research direction.

II. Linear Canonical Transform

The LCT of a signal $f(t)$ with the parameter matrix $\mathbf{A} = (a, b; c, d)$ is defined as [25]–[39]

$$F_{\mathbf{A}}(u) = \mathcal{L}^{\mathbf{A}}[f](u) = \begin{cases} \int_{-\infty}^{+\infty} f(t) K_{\mathbf{A}}(u,t)\mathrm{d}t, & b \neq 0 \\ \sqrt{d}\mathrm{e}^{\mathrm{j}\frac{cd}{2}u^2} f(du), & b = 0 \end{cases}, \tag{1}$$

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where the LCT basis takes

$$K_{\mathbf{A}}(u,t)=\frac{1}{\sqrt{\mathrm{j}2\pi b}}\mathrm{e}^{\mathrm{j}\left(\frac{d}{2b}u^2-\frac{1}{b}ut+\frac{a}{2b}t^2\right)},\tag{2}$$

and where LCT parameters a,b,c,d are real numbers satisfying $ad-bc=1$. The LCT is invertible and its inversion formula is given by

$$f(t)=\mathcal{L}^{\mathbf{A}^{-1}}[F_{\mathbf{A}}](t),\tag{3}$$

where $\mathbf{A}^{-1}=(d,-b;-c,a)$.

The variable u found in above formulae stands for the LCT frequency that is simplified as the linear canonical frequency. The LCT domain is thus simplified as the linear canonical domain accordingly.

As it is seen, the LCTs of $\mathbf{A}=(\cos\alpha,\sin\alpha;-\sin\alpha,\cos\alpha)$ and $\mathbf{A}=(0,1;-1,0)$ reduce to the FRFT and the FT, denoted by F_α and F, respectively.

It is obvious that the LCT of $b=0$ is essentially a scaling and chirp multiplication operations. Without loss of generality, we therefore focus merely on the LCT of $b\neq 0$. In that case, the relation $c=\frac{ad-1}{b}$ holds, implying that there exist only three free parameters a,b,d.

III. Closed-Form Cohen's Class Time-Frequency Representation in Linear Canonical Domains

There are a large number of WDs in linear canonical domains, for example, the AC type of WD (ACWD) [16], the BF type of WD (BFWD) [18], the CE type of WD (CEWD) [21], the ICF type of WD (ICFWD) [22], and the CICF type of WD (CICFWD) [23], [24]. The relationship between these general WDs was studied, and it is revealed that the CICFWD unifies all of them, and can be considered as the WD's closed-form representation in linear canonical domains [23], [24]. The WD is a generating distribution of the Cohen's class. It is then natural that the CICFCC is the main research object of the theory of Cohen's class in linear canonical domains, just because the CICFWD is the main research object of the theory of WD in linear canonical domains.

In this section, the definition of CICFCC is derived from the CICF type of LCT free parameters embedded approach that has been used to yield the CICFWD. Some essential properties of the CICFCC regarding those of the traditional Cohen's class are also deduced as a theoretical foundation for future practical applications.

A. Definition

Let the superscript $*$ denote complex conjugate, $F_{\mathbf{A}_1}$ and $F_{\mathbf{A}_2}$ denote LCTs of parameter matrices $\mathbf{A}_1=(a_1,b_1;c_1,d_1)$ and $\mathbf{A}_2=(a_2,b_2;c_2,d_2)$ respectively, and $K_{\mathbf{A}}$ denote LCT basis of the parameter matrix $\mathbf{A}=(a,b;c,d)$. The CICF type of LCT free parameters embedded approach originates from the idea which is to replace the traditional time domain instantaneous autocorrelation function $f\left(t+\frac{\tau}{2}\right)f^*\left(t-\frac{\tau}{2}\right)$ with the closed-form linear canonical domain instantaneous cross-correlation function $F_{\mathbf{A}_1}\left(u+\frac{\tau}{2}\right)F^*_{\mathbf{A}_2}\left(u-\frac{\tau}{2}\right)$ and substitute the FT basis $\mathrm{e}^{-\mathrm{j}\omega\tau}$ by the LCT one $K_{\mathbf{A}}(u,\tau)$. It follows that the definition of CICFCC associated with a kernel $\phi(\theta,\tau)$ is given by a treble parameter matrices class of bilinear integral transformation

$$\iiint_{-\infty}^{+\infty}F_{\mathbf{A}_1}\left(v+\frac{\tau}{2}\right)F^*_{\mathbf{A}_2}\left(v-\frac{\tau}{2}\right)\times\phi(\theta,\tau)K_{\mathbf{A}}(t,\theta)K_{\mathbf{A}}(u,\tau)K_{\mathbf{A}^{-1}}(\theta,v)\mathrm{d}v\mathrm{d}\tau\mathrm{d}\theta,\tag{4}$$

which is denoted by $\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,u)$.

The CICFCC of $\mathbf{A}_i=(\cos\alpha_i,\sin\alpha_i;-\sin\alpha_i,\cos\alpha_i)$ $(i=1,2)$ and $\mathbf{A}=(\cos\alpha,\sin\alpha;-\sin\alpha,\cos\alpha)$ reduces to the closed-form Cohen's class time-frequency representation in fractional domains (i.e., a kind of time-frequency domain provided by the FRFT)

$$\mathrm{CC}_f^{\alpha_1,\alpha_2,\alpha}(t,u)=\iiint_{-\infty}^{+\infty}F_{\alpha_1}\left(v+\frac{\tau}{2}\right)F^*_{\alpha_2}\left(v-\frac{\tau}{2}\right)\times\phi(\theta,\tau)K_{\alpha}(t,\theta)K_{\alpha}(u,\tau)K_{-\alpha}(\theta,v)\mathrm{d}v\mathrm{d}\tau\mathrm{d}\theta,\tag{5}$$

where K_α and $K_{-\alpha}$ denote FRFT basis operators.

It is evident that the CICFCC of $\mathbf{A}_i=(1,0;0,1)$ $(i=1,2)$ and $\mathbf{A}=(0,1;-1,0)$ reduces to the traditional one

$$\mathrm{CC}_f(t,\omega)=\iiint_{-\infty}^{+\infty}f\left(v+\frac{\tau}{2}\right)f^*\left(v-\frac{\tau}{2}\right)\times\phi(\theta,\tau)\mathrm{e}^{-\mathrm{j}t\theta}\mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{e}^{\mathrm{j}\theta v}\mathrm{d}v\mathrm{d}\tau\mathrm{d}\theta.\tag{6}$$

Moreover, see Table I that the CICFCC unifies the AC type of Cohen's class (ACCC), the BF type of Cohen's class (BFCC), the CE type of Cohen's class (CECC), and the ICF type of Cohen's class (ICFCC) for specific parameter matrix combinations, and then it can be seen as the Cohen's class's closed-form representation in linear canonical domains.

TABLE I
Some well-known Cohen's classes in linear canonical domains

$\begin{bmatrix}a_1 & b_1\\ c_1 & d_1\end{bmatrix}$	$\begin{bmatrix}a_2 & b_2\\ c_2 & d_2\end{bmatrix}$	$\begin{bmatrix}a & b\\ c & d\end{bmatrix}$	CICFCC
$\begin{bmatrix}a_1 & b_1\\ c_1 & d_1\end{bmatrix}$	$\begin{bmatrix}a_1 & b_1\\ c_1 & d_1\end{bmatrix}$	$\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$	ACCC
$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$	$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$	$\begin{bmatrix}a & b\\ c & d\end{bmatrix}$	BFCC
$\begin{bmatrix}a_1 & b_1\\ c_1 & d_1\end{bmatrix}$	$\begin{bmatrix}-a_1 & b_1\\ c_1 & -d_1\end{bmatrix}$	$\begin{bmatrix}\frac{d_1}{4} & -b_1\\ -c_1 & 4a_1\end{bmatrix}$	CECC
$\begin{bmatrix}a_1 & b_1\\ c_1 & d_1\end{bmatrix}$	$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$	$\begin{bmatrix}a & b\\ c & d\end{bmatrix}$	ICFCC

B. Essential Properties

In this part, we extend some essential properties (e.g., marginal distribution, energy conservation, unique reconstruction, Moyal formula, complex conjugate symmetry, time reversal symmetry, scaling property, time shift property, frequency shift property) of the traditional Cohen's class to those of the CICFCC, and show that how those general properties get reflected as constraints on LCT parameters and the kernel. We also propose a new CICFCC's property named as LCT invariance. Note that the proof of some of these properties is straightforward, just by using the CICFCC's definition and LCT's essential properties, and then it is omitted.

Marginal distribution.

The marginal distribution we discuss here is similar to the classical one in probability theory and statistics. It gives a distribution of various values of the linear canonical frequency u (the time t) variable without reference to the values of the time t (the linear canonical frequency u) variable. There are four types of CICFCC's marginal distributions including time, frequency, time delay, and frequency shift marginal distributions.

Time marginal distribution: The integration of CICFCC with respect to the linear canonical frequency u has a form

$$\begin{aligned}&\frac{1}{\sqrt{-\mathrm{j}2\pi b}}\int_{-\infty}^{+\infty}\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,u)\mathrm{e}^{-\mathrm{j}\frac{d}{2b}u^2}\mathrm{d}u\\&=\int_{-\infty}^{+\infty}F_{\mathbf{A}_1}(v)F_{\mathbf{A}_2}^*(v)\mathrm{e}^{\mathrm{j}\frac{d}{2b}(t^2-v^2)}\mathrm{d}v\\&\times\frac{1}{2\pi|b|}\int_{-\infty}^{+\infty}\mathrm{e}^{-\mathrm{j}\frac{1}{b}\theta(t-v)}\phi(\theta,0)\mathrm{d}\theta.\end{aligned}\tag{7}$$

If the kernel satisfies

$$\phi(\theta,0)=1,\tag{8}$$

it follows that the inner integral found in (7) can be calculated as

$$\int_{-\infty}^{+\infty}\mathrm{e}^{-\mathrm{j}\frac{1}{b}\theta(t-v)}\phi(\theta,0)\mathrm{d}\theta=2\pi|b|\delta(t-v),\tag{9}$$

where δ denotes Dirac delta operator, and then the formula (7) becomes

$$\frac{1}{\sqrt{-\mathrm{j}2\pi b}}\int_{-\infty}^{+\infty}\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,u)\mathrm{e}^{-\mathrm{j}\frac{d}{2b}u^2}\mathrm{d}u=F_{\mathbf{A}_1}(t)F_{\mathbf{A}_2}^*(t),\tag{10}$$

giving rise to the time marginal distribution of CICFCC.

Frequency marginal distribution: The integration of CICFCC with respect to the time t has a form

$$\begin{aligned}&\frac{\mathrm{e}^{-\mathrm{j}\frac{d+4a}{2b}u^2}}{\sqrt{-\mathrm{j}2\pi b}}\int_{-\infty}^{+\infty}\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,u)\mathrm{e}^{-\mathrm{j}\frac{d}{2b}t^2}\mathrm{d}t\\&=\frac{\mathrm{e}^{-\mathrm{j}\frac{2a}{b}u^2}}{2\pi|b|}\iint_{-\infty}^{+\infty}F_{\mathbf{A}_1}\left(v+\frac{\tau}{2}\right)F_{\mathbf{A}_2}^*\left(v-\frac{\tau}{2}\right)\\&\times\phi(0,\tau)\mathrm{e}^{\mathrm{j}\frac{a}{2b}\tau^2}\mathrm{e}^{-\mathrm{j}\frac{1}{b}u\tau}\mathrm{e}^{-\mathrm{j}\frac{d}{2b}v^2}\mathrm{d}v\mathrm{d}\tau.\end{aligned}\tag{11}$$

If the kernel satisfies

$$\phi(0,\tau)=\mathrm{e}^{-\mathrm{j}\frac{d+4a}{8b}\tau^2},\tag{12}$$

it follows from the change of variables $x=v+\frac{\tau}{2}$ and $y=v-\frac{\tau}{2}$ that

$$\begin{aligned}&\frac{\mathrm{e}^{-\mathrm{j}\frac{d+4a}{2b}u^2}}{\sqrt{-\mathrm{j}2\pi b}}\int_{-\infty}^{+\infty}\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,u)\mathrm{e}^{-\mathrm{j}\frac{d}{2b}t^2}\mathrm{d}t\\&=\int_{-\infty}^{+\infty}F_{\mathbf{A}_1}(x)K_{(-d/2,b;c,-2a)}(u,x)\mathrm{d}x\\&\times\left[\int_{-\infty}^{+\infty}F_{\mathbf{A}_2}(y)K_{(d/2,b;c,2a)}(u,y)\mathrm{d}y\right]^*.\end{aligned}\tag{13}$$

Using the additivity of LCT, the above formula turns into

$$\begin{aligned}&\frac{\mathrm{e}^{-\mathrm{j}\frac{d+4a}{2b}u^2}}{\sqrt{-\mathrm{j}2\pi b}}\int_{-\infty}^{+\infty}\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,u)\mathrm{e}^{-\mathrm{j}\frac{d}{2b}t^2}\mathrm{d}t\\&=F_{(-d/2,b;c,-2a)\mathbf{A}_1}(u)F_{(d/2,b;c,2a)\mathbf{A}_2}^*(u),\end{aligned}\tag{14}$$

which provides the frequency marginal distribution of CICFCC.

Time delay marginal distribution: The CICFCC of $t=0$ takes

$$\begin{aligned}&\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(0,u)\\&=\iint_{-\infty}^{+\infty}F_{\mathbf{A}_1}\left(v+\frac{\tau}{2}\right)F_{\mathbf{A}_2}^*\left(v-\frac{\tau}{2}\right)\mathrm{e}^{-\mathrm{j}\frac{d}{2b}v^2}K_{\mathbf{A}}(u,\tau)\mathrm{d}v\mathrm{d}\tau\\&\times\frac{1}{2\pi|b|}\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{j}\frac{1}{b}\theta v}\phi(\theta,\tau)\mathrm{d}\theta.\end{aligned}\tag{15}$$

Frequency shift marginal distribution: The CICFCC of $u=0$ takes

$$\begin{aligned}\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,0)=&\frac{1}{\sqrt{\mathrm{j}2\pi b}}\iint_{-\infty}^{+\infty}F_{\mathbf{A}_1}\left(v+\frac{\tau}{2}\right)F_{\mathbf{A}_2}^*\left(v-\frac{\tau}{2}\right)\\&\times\mathrm{e}^{\mathrm{j}\frac{d}{2b}(t^2-v^2)}\mathrm{e}^{\mathrm{j}\frac{a}{2b}\tau^2}\mathrm{d}v\mathrm{d}\tau\\&\times\frac{1}{2\pi|b|}\int_{-\infty}^{+\infty}\mathrm{e}^{-\mathrm{j}\frac{1}{b}\theta(t-v)}\phi(\theta,\tau)\mathrm{d}\theta.\end{aligned}\tag{16}$$

Energy conservation. The energy conservation is closely related to the marginal distribution. It demonstrates that a sum of a signal's CICFCC's various values of the time t and the linear canonical frequency u variables equals to the signal's energy. The CICFCC's energy conservation has three kinds of equivalent forms, those are, time, frequency, and time delay or frequency shift marginal distributions based energy conservation.

Time marginal distribution based energy conservation: Integrating on both sides of (10) for the time t yields

$$\begin{aligned}&\frac{1}{\sqrt{-\mathrm{j}2\pi b}}\iint_{-\infty}^{+\infty}\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_1,\mathbf{A}}(t,u)\mathrm{e}^{-\mathrm{j}\frac{d}{2b}u^2}\mathrm{d}t\mathrm{d}u\\&=\int_{-\infty}^{+\infty}|f(t)|^2\mathrm{d}t.\end{aligned}\tag{17}$$

Frequency marginal distribution based energy conservation: Integrating on both sides of (14) for the linear canonical frequency u yields

$$\frac{\mathrm{e}^{-\mathrm{j}\frac{d+4a}{2b}u^2}}{\sqrt{-\mathrm{j}2\pi b}}\iint_{-\infty}^{+\infty}\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,u)\mathrm{e}^{-\mathrm{j}\frac{d}{2b}t^2}\mathrm{d}t\mathrm{d}u = \int_{-\infty}^{+\infty}|f(t)|^2\mathrm{d}t, \tag{18}$$

where $(-d/2,b;c,-2a)\mathbf{A}_1=(d/2,b;c,2a)\mathbf{A}_2$.

Time delay or frequency shift marginal distribution based energy conservation: The CICFCC of $t=u=0$ takes

$$\begin{aligned}\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(0,0)=&\frac{1}{\sqrt{\mathrm{j}2\pi b}}\iint_{-\infty}^{+\infty}F_{\mathbf{A}_1}\left(v+\frac{\tau}{2}\right)F_{\mathbf{A}_2}^*\left(v-\frac{\tau}{2}\right)\\&\times\mathrm{e}^{-\mathrm{j}\frac{d}{2b}v^2}\mathrm{e}^{\mathrm{j}\frac{a}{2b}\tau^2}\mathrm{d}v\mathrm{d}\tau\\&\times\frac{1}{2\pi|b|}\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{j}\frac{1}{b}\theta v}\phi(\theta,\tau)\mathrm{d}\theta.\end{aligned} \tag{19}$$

If the kernel takes

$$\phi(\theta,\tau)=1 \tag{20}$$

satisfying

$$\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{j}\frac{1}{b}\theta v}\phi(\theta,\tau)\mathrm{d}\theta=2\pi|b|\delta(v), \tag{21}$$

the formula (19) becomes

$$\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(0,0)=\frac{2}{\sqrt{\mathrm{j}2\pi b}}\int_{-\infty}^{+\infty}F_{\mathbf{A}_1}(\tau)F_{\mathbf{A}_2}^*(-\tau)\mathrm{e}^{\mathrm{j}\frac{2a}{b}\tau^2}\mathrm{d}\tau. \tag{22}$$

Let $\mathbf{A}_2=-\mathbf{A}_1$ and $a=0$, it follows that

$$\frac{\mathrm{j}\sqrt{\mathrm{j}2\pi b}}{2}\mathrm{CC}_f^{\mathbf{A}_1,-\mathbf{A}_1,\mathbf{A}}(0,0)=\int_{-\infty}^{+\infty}|f(t)|^2\mathrm{d}t. \tag{23}$$

Unique reconstruction.

The unique reconstruction of a distribution reveals that a signal can be recovered by the distribution of it. Specifically, next we will show that there is a relationship between a signal and its CICFCC.

Using the inversion formula (3), the relation

$$\begin{aligned}&\int_{-\infty}^{+\infty}\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,u)K_{\mathbf{A}^{-1}}(\tau,u)\mathrm{d}u\\=&\int_{-\infty}^{+\infty}F_{\mathbf{A}_1}\left(v+\frac{\tau}{2}\right)F_{\mathbf{A}_2}^*\left(v-\frac{\tau}{2}\right)\mathrm{e}^{\mathrm{j}\frac{d}{2b}(t^2-v^2)}\mathrm{d}v\\&\times\frac{1}{2\pi|b|}\int_{-\infty}^{+\infty}\mathrm{e}^{-\mathrm{j}\frac{1}{b}\theta(t-v)}\phi(\theta,\tau)\mathrm{d}\theta\end{aligned} \tag{24}$$

holds. If the kernel is independent of the variable θ, i.e.,

$$\phi(\theta,\tau)=\varphi(\tau), \tag{25}$$

it follows that the inner integral found in (24) can be calculated as

$$\int_{-\infty}^{+\infty}\mathrm{e}^{-\mathrm{j}\frac{1}{b}\theta(t-v)}\phi(\theta,\tau)\mathrm{d}\theta=2\pi|b|\varphi(\tau)\delta(t-v), \tag{26}$$

and then the formula (24) turns into

$$\begin{aligned}&\int_{-\infty}^{+\infty}\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,u)K_{\mathbf{A}^{-1}}(\tau,u)\mathrm{d}u\\=&\varphi(\tau)F_{\mathbf{A}_1}\left(t+\frac{\tau}{2}\right)F_{\mathbf{A}_2}^*\left(t-\frac{\tau}{2}\right).\end{aligned} \tag{27}$$

Using again the inversion formula (3), there is a relation

$$\begin{aligned}f(\eta)=&\iint_{-\infty}^{+\infty}\frac{1}{\varphi^*(-t)F_{\mathbf{A}_1}^*(0)}\left[\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}\left(\frac{t}{2},u\right)\right]^*\\&\times K_{\mathbf{A}}(u,-t)K_{\mathbf{A}_2^{-1}}(\eta,t)\mathrm{d}u\mathrm{d}t\end{aligned} \tag{28}$$

for $t=-\frac{\tau}{2}$ or

$$\begin{aligned}f(\eta)=&\iint_{-\infty}^{+\infty}\frac{1}{\varphi(t)F_{\mathbf{A}_2}^*(0)}\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}\left(\frac{t}{2},u\right)\\&\times K_{\mathbf{A}^{-1}}(t,u)K_{\mathbf{A}_1^{-1}}(\eta,t)\mathrm{d}u\mathrm{d}t\end{aligned} \tag{29}$$

for $t=\frac{\tau}{2}$, implying that a signal can be reconstructed by its CICFCC uniquely.

Moyal formula.

The Moyal formula, which calculates an integration of the product of two signals' CICFCCs for the time t and the linear canonical frequency u, is a generalization of the energy conservation. Then, there is a formula

$$\begin{aligned}&\iint_{-\infty}^{+\infty}\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,u)\left[\mathrm{CC}_g^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,u)\right]^*\mathrm{d}t\mathrm{d}u\\=&\iiint_{-\infty}^{+\infty}F_{\mathbf{A}_1}\left(v+\frac{\tau}{2}\right)F_{\mathbf{A}_2}^*\left(v-\frac{\tau}{2}\right)\\&\times G_{\mathbf{A}_1}^*\left(\varepsilon+\frac{\tau}{2}\right)G_{\mathbf{A}_2}\left(\varepsilon-\frac{\tau}{2}\right)\mathrm{e}^{\mathrm{j}\frac{d}{2b}(\varepsilon^2-v^2)}\mathrm{d}\varepsilon\mathrm{d}v\mathrm{d}\tau\\&\times\frac{1}{2\pi|b|}\int_{-\infty}^{+\infty}\mathrm{e}^{-\mathrm{j}\frac{1}{b}\theta(\varepsilon-v)}\left|\phi(\theta,\tau)\right|^2\mathrm{d}\theta.\end{aligned} \tag{30}$$

If the kernel's module is 1, that is,

$$|\phi(\theta,\tau)|=1 \tag{31}$$

satisfying

$$\int_{-\infty}^{+\infty}\mathrm{e}^{-\mathrm{j}\frac{1}{b}\theta(\varepsilon-v)}\left|\phi(\theta,\tau)\right|^2\mathrm{d}\theta=2\pi|b|\delta(\varepsilon-v), \tag{32}$$

it follows from the change of variables $x=v+\frac{\tau}{2}$ and $y=v-\frac{\tau}{2}$ that

$$\begin{aligned}&\iint_{-\infty}^{+\infty}\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,u)\left[\mathrm{CC}_g^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,u)\right]^*\mathrm{d}t\mathrm{d}u\\=&\left|\int_{-\infty}^{+\infty}f(t)g^*(t)\mathrm{d}t\right|^2,\end{aligned} \tag{33}$$

which is the CICFCC's Moyal formula.

Complex conjugate symmetry.

The complex conjugate symmetry gives a relationship between a signal's CICFCC and the CICFCC of the complex conjugate signal, i.e.,

$$\begin{aligned}&\mathrm{CC}_{f^*}^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,u)\\=&\left[\mathrm{CC}_f^{(a_1,-b_1;-c_1,d_1),(a_2,-b_2;-c_2,d_2),(a,-b;-c,d)}(t,u)\right]^*\end{aligned} \tag{34}$$

for a real kernel $\phi^*(\theta,\tau)=\phi(\theta,\tau)$ or

$$\mathrm{CC}_{f^*}^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,u)=\mathrm{CC}_f^{(a_2,-b_2;-c_2,d_2),(a_1,-b_1;-c_1,d_1),\mathbf{A}}(t,-u) \tag{35}$$

for the variable τ symmetric kernel $\phi(\theta,-\tau)=\phi(\theta,\tau)$.

Time reversal symmetry.

The time reversal symmetry presents a relationship between a signal's CICFCC and the CICFCC of the time reversal signal, i.e.,

$$\mathrm{CC}_{f(-\cdot)}^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,u)=\left[\mathrm{CC}_f^{\mathbf{A}_2,\mathbf{A}_1,(a,-b;-c,d)}(-t,u)\right]^* \tag{36}$$

for the variable θ conjugate symmetric kernel $\phi^*(-\theta,\tau)=\phi(\theta,\tau)$ or

$$\mathrm{CC}_{f(-\cdot)}^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,u)=\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(-t,-u). \tag{37}$$

for the variables θ and τ symmetric kernel $\phi(-\theta,-\tau)=\phi(\theta,\tau)$.

Scaling property.

The scaling property establishes a relationship between a signal's CICFCC and the CICFCC of the scaled signal, i.e.,

$$\begin{aligned}&\mathrm{CC}_{f(\sigma\cdot)}^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,u)\\ =&\frac{1}{|\sigma|}\mathrm{CC}_f^{(a_1/\sigma,b_1\sigma;c_1/\sigma,d_1\sigma),(a_2/\sigma,b_2\sigma;c_2/\sigma,d_2\sigma),\mathbf{A}}(t,u).\end{aligned} \tag{38}$$

Time shift property.

The time shift property builds a relationship between a signal's CICFCC and the CICFCC of the time shift signal. Then, there is a formula

$$\begin{aligned}&\mathrm{CC}_{f(\cdot-t_0)}^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,u)\\ =&\iiint_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{j}c_1\left[t_0\left(v+\frac{\tau}{2}\right)-\frac{a_1}{2}t_0^2\right]}F_{\mathbf{A}_1}\left(v+\frac{\tau}{2}-a_1t_0\right)\\ &\times\mathrm{e}^{-\mathrm{j}c_2\left[t_0\left(v-\frac{\tau}{2}\right)-\frac{a_2}{2}t_0^2\right]}F_{\mathbf{A}_2}^*\left(v-\frac{\tau}{2}-a_2t_0\right)\\ &\times\phi(\theta,\tau)K_{\mathbf{A}}(t,\theta)K_{\mathbf{A}}(u,\tau)K_{\mathbf{A}^{-1}}(\theta,v)\mathrm{d}v\mathrm{d}\tau\mathrm{d}\theta.\end{aligned} \tag{39}$$

Let $a_1=a_2$, it follows from the the change of variables $\varepsilon=v-a_1t_0$ and $\xi=\theta+[(c_1-c_2)b-a_1d]t_0$ that

$$\begin{aligned}&\mathrm{CC}_{f(\cdot-t_0)}^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,u)\\ =&\mathrm{e}^{-\mathrm{j}\frac{bd(c_1+c_2)^2+4a_1(c_1-c_2)}{8}t_0^2}\mathrm{e}^{\mathrm{j}\frac{2(c_1-c_2)t+d(c_1+c_2)u}{2}t_0}\\ &\times\iiint_{-\infty}^{+\infty}F_{\mathbf{A}_1}\left(\varepsilon+\frac{\tau}{2}\right)F_{\mathbf{A}_2}^*\left(\varepsilon-\frac{\tau}{2}\right)\\ &\times\phi(\xi-[(c_1-c_2)b-a_1d]t_0,\tau)K_{\mathbf{A}}(t-a_1t_0,\xi)\\ &\times K_{\mathbf{A}}\left(u-\frac{b(c_1+c_2)}{2}t_0,\tau\right)K_{\mathbf{A}^{-1}}(\xi,\varepsilon)\mathrm{d}\varepsilon\mathrm{d}\tau\mathrm{d}\xi.\end{aligned} \tag{40}$$

(i) If the kernel is shift separable for the variable θ, that is,

$$\phi(\theta-\eta,\tau)=\phi(\theta,\tau)\psi(\eta), \tag{41}$$

the formula (40) becomes

$$\begin{aligned}&\mathrm{CC}_{f(\cdot-t_0)}^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,u)\\ =&\mathrm{e}^{-\mathrm{j}\frac{bd(c_1+c_2)^2+4a_1(c_1-c_2)}{8}t_0^2}\mathrm{e}^{\mathrm{j}\frac{2(c_1-c_2)t+d(c_1+c_2)u}{2}t_0}\\ &\times\psi([(c_1-c_2)b-a_1d]t_0)\\ &\times\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}\left(t-a_1t_0,u-\frac{b(c_1+c_2)}{2}t_0\right).\end{aligned} \tag{42}$$

(ii) If $(c_1-c_2)b=a_1d$, the formula (40) becomes

$$\begin{aligned}&\mathrm{CC}_{f(\cdot-t_0)}^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,u)\\ =&\mathrm{e}^{-\mathrm{j}\frac{bd(c_1+c_2)^2+4a_1(c_1-c_2)}{8}t_0^2}\mathrm{e}^{\mathrm{j}\frac{2(c_1-c_2)t+d(c_1+c_2)u}{2}t_0}\\ &\times\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}\left(t-a_1t_0,u-\frac{b(c_1+c_2)}{2}t_0\right).\end{aligned} \tag{43}$$

The above two formulae provide two types of time shift properties of CICFCC.

Frequency shift property.

The frequency shift property demonstrates a relationship between a signal's CICFCC and the CICFCC of the frequency shift signal. Similar to the time shift properties derived, it is easy to obtain two kinds of frequency shift properties of CICFCC, those are,

$$\begin{aligned}&\mathrm{CC}_{f(\cdot)\mathrm{e}^{\mathrm{j}u_0\cdot}}^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,u)\\ =&\mathrm{e}^{-\mathrm{j}\frac{bd(d_1+d_2)^2+4b_1(d_1-d_2)}{8}u_0^2}\mathrm{e}^{\mathrm{j}\frac{2(d_1-d_2)t+d(d_1+d_2)u}{2}u_0}\\ &\times\psi([(d_1-d_2)b-b_1d]u_0)\\ &\times\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}\left(t-b_1u_0,u-\frac{b(d_1+d_2)}{2}u_0\right),\end{aligned} \tag{44}$$

where $b_1=b_2$ and the kernel satisfies (41), and

$$\begin{aligned}&\mathrm{CC}_{f(\cdot)\mathrm{e}^{\mathrm{j}u_0\cdot}}^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,u)\\ =&\mathrm{e}^{-\mathrm{j}\frac{bd(d_1+d_2)^2+4b_1(d_1-d_2)}{8}u_0^2}\mathrm{e}^{\mathrm{j}\frac{2(d_1-d_2)t+d(d_1+d_2)u}{2}u_0}\\ &\times\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}\left(t-b_1u_0,u-\frac{b(d_1+d_2)}{2}u_0\right),\end{aligned} \tag{45}$$

where $b_1=b_2$ and $(d_1-d_2)b=b_1d$.

LCT invariance.

Different from the above traditional properties, the LCT invariance is a new property of the CICFCC. It reveals an equivalent relationship between a signal's CICFCC and the CICFCC of the signal's LCT, i.e.,

$$\mathrm{CC}_{F_{\mathbf{A}_0}}^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,u)=\mathrm{CC}_f^{\mathbf{A}_1\mathbf{A}_0,\mathbf{A}_2\mathbf{A}_0,\mathbf{A}}(t,u). \tag{46}$$

See Table II for a summary of the derived properties of CICFCC. Note that there are constraints on LCT parameters or the kernel for the CICFCC's time marginal distribution, frequency marginal distribution, energy conservation, unique reconstruction, Moyal formula, complex conjugate symmetry, time reversal symmetry, time shift property, and frequency shift property. Moreover, LCT parameters c_1, c_2, and c can be replaced by LCT free parameters as $c_1=\frac{a_1d_1-1}{b_1}$, $c_2=\frac{a_2d_2-1}{b_2}$, and $c=\frac{ad-1}{b}$, respectively.

TABLE II
ESSENTIAL PROPERTIES OF CICFCC

Property	Mathematical formula
Time marginal distribution[a]	$\frac{1}{\sqrt{-\mathrm{j}2\pi b}}\int_{-\infty}^{+\infty}\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,u)\mathrm{e}^{-\mathrm{j}\frac{d}{2b}u^2}\mathrm{d}u = F_{\mathbf{A}_1}(t)F^*_{\mathbf{A}_2}(t)$
Frequency marginal distribution[b]	$\frac{\mathrm{e}^{-\mathrm{j}\frac{d+4a}{2b}u^2}}{\sqrt{-\mathrm{j}2\pi b}}\int_{-\infty}^{+\infty}\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,u)\mathrm{e}^{-\mathrm{j}\frac{d}{2b}t^2}\mathrm{d}t$ $= F_{(-d/2,b;c,-2a)\mathbf{A}_1}(u)F^*_{(d/2,b;c,2a)\mathbf{A}_2}(u)$
Time delay marginal distribution	$\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(0,u) = \iint_{-\infty}^{+\infty} F_{\mathbf{A}_1}\left(v+\frac{\tau}{2}\right)F^*_{\mathbf{A}_2}\left(v-\frac{\tau}{2}\right)\mathrm{e}^{-\mathrm{j}\frac{d}{2b}v^2}K_{\mathbf{A}}(u,\tau)\mathrm{d}v\mathrm{d}\tau$ $\times\frac{1}{2\pi\lvert b\rvert}\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{j}\frac{1}{b}\theta v}\phi(\theta,\tau)\mathrm{d}\theta$
Frequency shift marginal distribution	$\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,0) = \frac{1}{\sqrt{\mathrm{j}2\pi b}}\iint_{-\infty}^{+\infty} F_{\mathbf{A}_1}\left(v+\frac{\tau}{2}\right)F^*_{\mathbf{A}_2}\left(v-\frac{\tau}{2}\right)\mathrm{e}^{\mathrm{j}\frac{d}{2b}(t^2-v^2)}\mathrm{e}^{\mathrm{j}\frac{a}{2b}\tau^2}\mathrm{d}v\mathrm{d}\tau$ $\times\frac{1}{2\pi\lvert b\rvert}\int_{-\infty}^{+\infty}\mathrm{e}^{-\mathrm{j}\frac{1}{b}\theta(t-v)}\phi(\theta,\tau)\mathrm{d}\theta$
Time marginal distribution based energy conservation[c]	$\frac{1}{\sqrt{-\mathrm{j}2\pi b}}\iint_{-\infty}^{+\infty}\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_1,\mathbf{A}}(t,u)\mathrm{e}^{-\mathrm{j}\frac{d}{2b}u^2}\mathrm{d}t\mathrm{d}u = \int_{-\infty}^{+\infty}\lvert f(t)\rvert^2\mathrm{d}t$
Frequency marginal distribution based energy conservation[d]	$\frac{\mathrm{e}^{-\mathrm{j}\frac{d+4a}{2b}u^2}}{\sqrt{-\mathrm{j}2\pi b}}\iint_{-\infty}^{+\infty}\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,u)\mathrm{e}^{-\mathrm{j}\frac{d}{2b}t^2}\mathrm{d}t\mathrm{d}u = \int_{-\infty}^{+\infty}\lvert f(t)\rvert^2\mathrm{d}t$
Time delay or frequency shift marginal distribution based energy conservation[e]	$\frac{\mathrm{j}\sqrt{\mathrm{j}2\pi b}}{2}\mathrm{CC}_f^{\mathbf{A}_1,-\mathbf{A}_1,\mathbf{A}}(0,0) = \int_{-\infty}^{+\infty}\lvert f(t)\rvert^2\mathrm{d}t$
Unique reconstruction[f]	$f(\eta) = \iint_{-\infty}^{+\infty}\frac{1}{\varphi^*(-t)F^*_{\mathbf{A}_1}(0)}\left[\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}\left(\frac{t}{2},u\right)\right]^* K_{\mathbf{A}}(u,-t)K_{\mathbf{A}_2^{-1}}(\eta,t)\mathrm{d}u\mathrm{d}t,$ $f(\eta) = \iint_{-\infty}^{+\infty}\frac{1}{\varphi(t)F^*_{\mathbf{A}_2}(0)}\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}\left(\frac{t}{2},u\right)K_{\mathbf{A}^{-1}}(t,u)K_{\mathbf{A}_1^{-1}}(\eta,t)\mathrm{d}u\mathrm{d}t$
Moyal formula[g]	$\iint_{-\infty}^{+\infty}\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,u)\left[\mathrm{CC}_g^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(t,u)\right]^*\mathrm{d}t\mathrm{d}u = \left\lvert\int_{-\infty}^{+\infty}f(t)g^*(t)\mathrm{d}t\right\rvert^2$
Complex conjugate symmetry[h] (CICFCC of $f^*(t)$)	$\left[\mathrm{CC}_f^{(a_1,-b_1;-c_1,d_1),(a_2,-b_2;-c_2,d_2),(a,-b;-c,d)}(t,u)\right]^*,$ $\mathrm{CC}_f^{(a_2,-b_2;-c_2,d_2),(a_1,-b_1;-c_1,d_1),\mathbf{A}}(t,-u)$
Time reversal symmetry[i] (CICFCC of $f(-t)$)	$\left[\mathrm{CC}_f^{\mathbf{A}_2,\mathbf{A}_1,(a,-b;-c,d)}(-t,u)\right]^*, \mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}(-t,-u)$
Scaling property (CICFCC of $f(\sigma t),\ \sigma\neq 0$)	$\frac{1}{\lvert\sigma\rvert}\mathrm{CC}_f^{(a_1/\sigma,b_1\sigma;c_1/\sigma,d_1\sigma),(a_2/\sigma,b_2\sigma;c_2/\sigma,d_2\sigma),\mathbf{A}}(t,u)$
Time shift property[j] (CICFCC of $f(t-t_0)$)	$\mathrm{e}^{-\mathrm{j}\frac{bd(c_1+c_2)^2+4a_1(c_1-c_2)}{8}t_0^2}\mathrm{e}^{\mathrm{j}\frac{2(c_1-c_2)t+d(c_1+c_2)u}{2}t_0}$ $\times\psi([(c_1-c_2)b-a_1d]t_0)\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}\left(t-a_1t_0, u-\frac{b(c_1+c_2)}{2}t_0\right),$ $\mathrm{e}^{-\mathrm{j}\frac{bd(c_1+c_2)^2+4a_1(c_1-c_2)}{8}t_0^2}\mathrm{e}^{\mathrm{j}\frac{2(c_1-c_2)t+d(c_1+c_2)u}{2}t_0}$ $\times\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}\left(t-a_1t_0, u-\frac{b(c_1+c_2)}{2}t_0\right)$
Frequency shift property[k] (CICFCC of $f(t)\mathrm{e}^{\mathrm{j}u_0t}$)	$\mathrm{e}^{-\mathrm{j}\frac{bd(d_1+d_2)^2+4b_1(d_1-d_2)}{8}u_0^2}\mathrm{e}^{\mathrm{j}\frac{2(d_1-d_2)t+d(d_1+d_2)u}{2}u_0}$ $\times\psi([(d_1-d_2)b-b_1d]u_0)\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}\left(t-b_1u_0, u-\frac{b(d_1+d_2)}{2}u_0\right),$ $\mathrm{e}^{-\mathrm{j}\frac{bd(d_1+d_2)^2+4b_1(d_1-d_2)}{8}u_0^2}\mathrm{e}^{\mathrm{j}\frac{2(d_1-d_2)t+d(d_1+d_2)u}{2}u_0}$ $\times\mathrm{CC}_f^{\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}}\left(t-b_1u_0, u-\frac{b(d_1+d_2)}{2}u_0\right)$
LCT invariance (CICFCC of $F_{\mathbf{A}_0}$)	$\mathrm{CC}_f^{\mathbf{A}_1\mathbf{A}_0,\mathbf{A}_2\mathbf{A}_0,\mathbf{A}}(t,u)$

[a] Constraints on the kernel: $\phi(\theta,0)=1$
[b] Constraints on the kernel: $\phi(0,\tau)=\mathrm{e}^{-\mathrm{j}\frac{d+4a}{8b}\tau^2}$
[c] Constraints on LCT parameters and the kernel: $\mathbf{A}_2=\mathbf{A}_1$ and $\phi(\theta,0)=1$
[d] Constraints on LCT parameters and the kernel: $(-d/2,b;c,-2a)\mathbf{A}_1=(d/2,b;c,2a)\mathbf{A}_2$ and $\phi(0,\tau)=\mathrm{e}^{-\mathrm{j}\frac{d+4a}{8b}\tau^2}$
[e] Constraints on LCT parameters and the kernel: $\mathbf{A}_2=-\mathbf{A}_1$, $a=0$, and $\phi(\theta,\tau)=1$
[f] Constraints on the kernel: $\phi(\theta,\tau)=\varphi(\tau)$
[g] Constraints on the kernel: $\lvert\phi(\theta,\tau)\rvert=1$
[h] Constraints on the kernel: $\phi^*(\theta,\tau)=\phi(\theta,\tau)$ or $\phi(\theta,-\tau)=\phi(\theta,\tau)$
[i] Constraints on the kernel: $\phi^*(-\theta,\tau)=\phi(\theta,\tau)$ or $\phi(-\theta,-\tau)=\phi(\theta,\tau)$
[j] Constraints on LCT parameters or the kernel: $a_1=a_2$ and $\phi(\theta-\eta,\tau)=\phi(\theta,\tau)\psi(\eta)$ or $(c_1-c_2)b=a_1d$
[k] Constraints on LCT parameters or the kernel: $b_1=b_2$ and $\phi(\theta-\eta,\tau)=\phi(\theta,\tau)\psi(\eta)$ or $(d_1-d_2)b=b_1d$

IV. Potential Applications

The CICFCC, which combines the traditional Cohen's class with the LCT, is expected to be of efficient path for non-stationary signal representation flexibility enhancement, providing a certain amount of freedom embedded signal representation tool based solution for non-stationary signal separation and detection.

A. Non-Stationary Signal Separation

The CICF type of LCT free parameters embedded approach could possibly meet the inherent requirement of signal representation flexibility enhancement in non-stationary signal separation. In other word, LCT free parameters embedded in the CICFCC is expected to enable it to have a decoupling property in the separation of non-stationary signals, indicating that there would be a causation between LCT free parameters and the decoupling property. To model this causality mathematically, it becomes therefore interesting to explore the principle of non-stationary signal decoupling triggered by LCT free parameters.

It is well-known that there is an affine transformation relation between the LCT and the traditional Cohen's class, giving rise to the LCT an affine nature in the time-frequency plane. The affine equivalence relation obtained is thus the mathematical theory foundation of the designing of filter in linear canonical domains and the inherent requirement of non-stationary signal decoupling. Therefore, a qualitative analysis to affine equivalence relation between the LCT and the CICFCC is efficient enough to yield the decoupling parameters in linear canonical domains.

B. Non-Stationary Signal Detection

The CICF type of LCT free parameters embedded approach could possibly meet the inherent requirement of signal representation flexibility enhancement in non-stationary signal detection. In other word, LCT free parameters embedded in the CICFCC is expected to enable it to have an improved performance in the detection of non-stationary signals, implying that there would be a causation between LCT free parameters and the detection performance. To establish this causality a mathematical modeling, it is therefore meaningful to disclose the intrinsic mechanism of non-stationary signal detection performance improvement triggered by LCT free parameters.

The theoretical basis of detection performance evaluation is the output signal-to-noise ratio (SNR), the output SNR inequality or the optimal LCT free parameters corresponding to the maximum output SNR are thus the mathematical theory foundation of the output SNR improvement and the inherent requirement of non-stationary signal detection performance improvement. Therefore, a qualitative analysis to output SNR inequality between the CICFCC and the traditional one or to the CICFCC's output SNR optimization model is efficient enough to yield the desirable LCT free parameters.

V. Conclusions

A kind of Cohen's class time-frequency representation associated with the LCT is proposed through the CICF type of LCT free parameters embedded approach, and then is simplified as the CICFCC. It turns out that the CICFCC is the main research object of the theory of Cohen's class in linear canonical domains as it unifies some well-known linear canonical domains Cohen's classes including the ACCC, BFCC, CECC and ICFCC, and can be regarded as the Cohen's class's closed-form representation in linear canonical domains. Some important and useful properties of the CICFCC are then obtained, such as marginal distribution, energy conservation, unique reconstruction, Moyal formula, complex conjugate symmetry, time reversal symmetry, scaling property, time shift property, frequency shift property, and LCT invariance. It is shown that most of those properties (e.g., time marginal distribution, frequency marginal distribution, energy conservation, unique reconstruction, Moyal formula, complex conjugate symmetry, time reversal symmetry, time shift property, and frequency shift property) get reflected as constraints on LCT parameters or the kernel. The CICFCC's applications in non-stationary signal separation and detection are also investigated, resulting in some key details worthy of further elaborating, such as the affine equivalence relation and the output SNR inequality or optimization model.

The future work will be focused on both theory and application aspects: (i) refine some related theories of the CICFCC (discretization, convolution, sampling, uncertainty, etc.); (ii) develop the CICFCC's applications not only in non-stationary signal separation and detection but also in non-stationary signal classification, recognition, estimation, tracking, etc.

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