Nonuniform fast linear canonical transform

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***Abstract*—The linear canonical transform (LCT) is a generalized form of the Fourier transformation. It has been shown to be one of the most powerful tools in applied mathematics, signal processing and optics fields. The aim of this paper is to present a nonuniform fast linear canonical transform (NFLCT), which emerges in many areas of physics and engineering. The proposed algorithm generalizes the fast linear canonical transform to the case of non-integer frequencies on the interval $[-b\pi, b\pi]$. The algorithm requires $O(N \log N + N \log(1/\epsilon))$ arithmetic operations where ϵ is the precision of computations and N is the number of nodes. The efficiency of the approach is illustrated by simulations.**

I. Introduction

Linear canonical transform (LCT) is a four-parameter class of linear integral transform [1], [2], which includes many special cases, such as, the Fourier transform (FT), the fractional Fourier transform (FRFT), the Fresnel transform, the Lorentz transform and scaling operations. It has been become a popular analytical tool in the study of optic and signal processing.

The numerical approximation of the LCT is importent in many signal processing applications. After the continuous LCT has been introduced, the definition and implementation of the discrete linear canonical transform (DLCT) have been widely considered by many researchers [3]–[10]. The existing discrete algorithms are required that both input and output data are uniform sampling [7]–[12]. In some applications, however, such as, radar signal processing, SAR imaging system, the data are often non-uniform. Therefore, it is necessary to research the nonuniform fast linear canonical transform. In this paper, we present a algorithm for computing nonuniform DLCT based on interpolation approximation.

The rest of the paper is organized as follows: In Section II, the related preliminaries are presented which are used in the design of the algorithm. In Section III, we give the relevant facts from approximation theory which are the principal tool in this paper. Some numerical examples are presented in Section IV to illustrate the performance of the schemes. Finally, conclusions are drawn in Section V.

II. Preliminaries

A. Linear canonical transform

The linear canonical transform (LCT) with matrix $A = [a, b; c, d] \in SL(2, R)$ of a square-integrable signal or function $x(t)$ is

$$F^A[x(t)](u) = \begin{cases} \frac{1}{\sqrt{i2\pi b}} \int_{-\infty}^{\infty} x(t) e^{i\frac{a}{2b}t^2 + i\frac{d}{2b}u^2 - i\frac{u}{b}t} \mathrm{d}t, & b \neq 0 \\ \sqrt{d} e^{i\frac{cd}{2}u^2} x(du), & b = 0 \end{cases} \tag{1}$$

where a, b, c, d are real and independent of t and u, $SL(2, R)$ is a special linear group of two-dimensional real space. The $b = 0$ corresponds to a simple chirp multiplication operation. Therefore, we only focus on the LCT for $b \neq 0$ and assume without loss of generality that $b > 0$. The inverse LCT is obtained by the LCT with matrix $A^{-1} = [d, -c; -b, a]$, i.e.

$$x(t) = \frac{1}{\sqrt{-i2\pi b}} \int_{-\infty}^{\infty} F^A[x(t)](u) e^{-i\frac{a}{2b}t^2 - i\frac{d}{2b}u^2 + i\frac{u}{b}t} \mathrm{d}u. \tag{2}$$

Note that $a = d = 0$, $b = 1$,$c = -1$, the LCT reduces to FT, and converts into the FRFT when $a = d = \cos\theta$, $b = \sin\theta$, $c = -\sin\theta$. One can refer to [2] for a detailed description of the relations between the LCT and its special cases.

B. Linear canonical series

The linear canonical series (LCS) is a generalized form of Fourier series (FS), which can reveal the mixed time and frequency components of signals. The basis signal of LCS is defined as [13]

$$\varphi_{A,n}(t) = \sqrt{\frac{i}{T}} e^{-i\frac{a}{2b}t^2 - i\frac{d}{2b}(n2\pi b/T)^2 + i\frac{t}{b}(n2\pi b/T)} \tag{3}$$

The sequence $\{\cdots, \varphi_{A,-1}(t), \varphi_{A,0}(t), \varphi_{A,1}(t), \cdots\}$ constructs an orthonormal basis. The LCS expansion of signal $x(t)$ with time width T can be written as

$$\begin{aligned} x(t) &= \sum_{n=-\infty}^{\infty} C_{A,n} \varphi_{A,n}(t) \\ &= \sum_{n=-\infty}^{\infty} C_{A,n} \sqrt{\frac{i}{T}} e^{-i\frac{a}{2b}t^2 - i\frac{d}{2b}(n2\pi b/T)^2 + i\frac{t}{b}(n2\pi b/T)} \end{aligned} \tag{4}$$

where $t \in [-T/2, T/2]$ and $C_{A,n}$ are called LCS expansion coefficients with the parameter matrix A. The relationship between LCS and LCT is that the LCS expansion coefficients are the sampled values of LCT, as

$$C_{A,n} = \sqrt{\frac{2\pi b}{T}} F^A[x(t)](n\frac{2\pi b}{T}) \tag{5}$$

III. Main results

A. Statement of the problem

In this section, we will consider the nonuniform discrete linear canonical transform (NDLCT), which is defined as following

$$g_j = G(\mathbf{g})_j = \sum_{k=-N/2}^{N/2-1} \beta_k e^{-i\frac{a}{2b}t_j^2+i\frac{kt_j}{b}-i\frac{d}{2b}k^2} \tag{6}$$

where β_k is input data which can be uniform or nonuniform, g_j is output data. For the efficient application of the transformation described by Eq. (6) , the following two steps are necessary:

- to approximate each $e^{-ia/(2b)t^2+iu_mt/b-id/2b(u_m)^2}$ in term of a q-term linear canonical series;
- to approximate the value of a linear canonical series at each t_n in terms of values at the nearest q uniformly-spaced nodes.

In the following, we present the derivation of nonuniform fast linear canonical transform.

B. Derivation of the nonuniform fast linear canonical transform

The principal tool of this paper is that detailed analysis of linear canonical series of functions $\phi : [-b\pi, b\pi] \to \mathbf{C}$ given by the formula

$$\phi(t) = e^{-\alpha t^2} e^{-i\frac{a}{2b}t^2+i\frac{\beta t}{b}-i\frac{d}{2b}\beta^2} \tag{7}$$

where $\alpha > 1/2$ and β are real numbers. We present this analysis in theorems of this subsection.

Theorem 1: Let $\alpha > 1/2$, β are real numbers, then, for any $t \in (-b\pi, b\pi)$, we can obtain

$$\begin{aligned}&\left|\phi(t) - \sum_{k=-\infty}^{\infty} \rho_{A,k}\sqrt{\frac{i}{2b\pi}} e^{-i\frac{a}{2b}t^2+i\frac{kt}{b}-i\frac{d}{2b}k^2}\right| \\ &< e^{-\alpha\pi^2}\left[(1+\frac{1}{9b^2})(\alpha+3\alpha b+8b)+\frac{1}{6b}\right]\end{aligned} \tag{8}$$

where $\phi(t) = e^{-\alpha t^2} e^{-i\frac{a}{2b}t^2+i\frac{\beta t}{b}-i\frac{d}{2b}\beta^2}$, $A = [a, b; c, d]$, $k = -\infty, \cdots, \infty$

$$\rho_{A,k} = \sqrt{\frac{-i}{2b\pi}} e^{i\frac{d}{2b}(k^2-\beta^2)} \sqrt{\frac{\pi}{\alpha}} e^{-\frac{(\beta-k)^2}{4b^2\alpha}} \tag{9}$$

Proof: The kth linear canonical series coefficient for $\phi(t)$ is denoted by $\sigma_{A,k}$, so that for $t \in (-b\pi, b\pi)$,

$$\phi(t) = \sqrt{\frac{i}{2b\pi}} \sum_{k=-\infty}^{\infty} \sigma_{A,k} e^{-i\frac{a}{2b}t^2+i\frac{kt}{b}-i\frac{d}{2b}k^2} \tag{10}$$

where $\sigma_{A,k} = \sqrt{\frac{-i}{2b\pi}} \int_{-b\pi}^{b\pi} e^{i\frac{a}{2b}t^2-i\frac{kt}{b}+i\frac{d}{2b}k^2} \phi(t)dt$

For any real $\alpha > 0$ and complex z, we have the following equality

$$\int_{-\infty}^{\infty} e^{-\alpha t^2} e^{zt} dt = \sqrt{\frac{\pi}{\alpha}} e^{z^2/4\alpha} \tag{11}$$

thus

$$\begin{aligned}\sigma_{A,k} &= \sqrt{\frac{-i}{2b\pi}} e^{i\frac{d}{2b}(k^2-\beta^2)} \left[\int_{-\infty}^{\infty} e^{-\alpha t^2} e^{i\frac{(\beta-k)t}{b}} dt\right. \\ &\quad \left. - \int_{-\infty}^{-b\pi} e^{-\alpha t^2} e^{i\frac{(\beta-k)t}{b}} dt - \int_{b\pi}^{\infty} e^{-\alpha t^2} e^{i\frac{(\beta-k)t}{b}} dt\right] \\ &= \sqrt{\frac{-i}{2b\pi}} e^{i\frac{b}{2b}(k^2-\beta^2)} \sqrt{\frac{\pi}{\alpha}} e^{-\frac{(\beta-k)^2}{4b^2\alpha}} + \sqrt{\frac{-i}{2b\pi}} e^{i\frac{d}{2b}(k^2-\beta^2)} \\ &\quad \times \left[\int_{\infty}^{b\pi} e^{-\alpha t^2} e^{-i\frac{(\beta-k)t}{b}} dt - \int_{b\pi}^{\infty} e^{-\alpha t^2} e^{i\frac{(\beta-k)t}{b}} dt\right] \\ &= \rho_{A,k} - \sqrt{\frac{-i}{2b\pi}} e^{i\frac{d(k^2-\beta^2)}{b2}} 2\int_{b\pi}^{\infty} e^{-\alpha t^2} \cos[\frac{(\beta-k)t}{b}]dt\end{aligned} \tag{12}$$

Rearranging Eq.(10), we obtain

$$\sigma_{A,k} - \rho_{A,k} = -\sqrt{\frac{-i}{2b\pi}} e^{i\frac{d(k^2-\beta^2)}{2b}} 2\int_{b\pi}^{\infty} e^{-\alpha t^2} \cos[\frac{(\beta-k)t}{b}]dt \tag{13}$$

Owing to

$$\begin{aligned}&\left|2\int_{b\pi}^{\infty} e^{-\alpha t^2}\cos[(\beta-k)t/b]dt + e^{-b^2\alpha\pi^2}\int_{-b\pi}^{b\pi} e^{i\frac{(\beta-k)t}{b}}dt\right| \\ &\le 2\int_{b\pi}^{\infty} e^{-\alpha t^2}dt + 2b\pi e^{-\alpha b^2\pi^2} \\ &< 2b\pi e^{-\alpha b^2\pi^2}(\frac{1}{2b^2\alpha\pi^2}+1) < 2b\pi e^{-\alpha b^2\pi^2}(\frac{1}{b^2\pi^2}+1)\end{aligned} \tag{14}$$

and integrating by parts, we have

$$\begin{aligned}&2\int_{\pi}^{\infty} e^{-\alpha t^2}\cos[(\beta-k)t]dt \\ &= -\frac{2b}{\beta-k} e^{-\alpha b^2\pi^2} \sin[(\beta-k)\pi] \\ &+ \frac{4\alpha}{\beta-k}\int_{b\pi}^{\infty} te^{-\alpha t^2}\sin[(\beta-k)t/b]dt\end{aligned} \tag{15}$$

After rearranging the terms in Eq.(15) and integrating by parts again, we obtain

$$\begin{aligned}&\left|2\int_{\pi}^{\infty} e^{-\alpha t^2}\cos[(\beta-k)t]dt + \frac{2be^{-\alpha b^2\pi^2}}{\beta-k}\sin[(\beta-k)\pi]\right| \\ &= \left|\frac{4b\alpha}{\beta-k}\int_{b\pi}^{\infty} te^{-\alpha t^2}\sin[(\beta-k)t/b]dt\right| \\ &\le \frac{4b^2\alpha}{(\beta-k)^2}\left(b\pi e^{-\alpha b^2\pi^2} + \int_{b\pi}^{\infty} e^{-\alpha t^2}dt\right. \\ &+ \left.\int_{b\pi}^{\infty} t\cdot 2\alpha te^{-\alpha t^2}dt\right) \\ &< \frac{4b^2\alpha}{(\beta-k)^2}\left(2b\pi e^{-\alpha b^2\pi^2} + 2\int_{b\pi}^{\infty} e^{-\alpha t^2}dt\right) \\ &< \frac{4b^2\alpha e^{-\alpha b^2\pi^2}}{(\beta-k)^2}(2b\pi + \frac{2}{2\alpha b\pi}) \\ &< \frac{8b^3\pi\alpha e^{-\alpha b^2\pi^2}}{(\beta-k)^2}(1+\frac{1}{b^2\pi^2})\end{aligned} \tag{16}$$

Therefore, we can obtain the following inequalities

$$\left|2\int_{\pi}^{\infty} e^{-\alpha t^2}\cos[(\beta-k)t]dt + e^{-\alpha\pi^2}\int_{-b\pi}^{b\pi} e^{i(\beta-k)t/b}dt\right| < \frac{8b^3\pi\alpha e^{-\alpha b^2\pi^2}}{(\beta-k)^2}(1+\frac{1}{b^2\pi^2}) \tag{17}$$

Due to Eq.(14) and (17), we have the inequalities

$$\begin{aligned}&\left|\sigma_{A,k}-\rho_{A,k}-\sqrt{\frac{-i}{2b\pi}}e^{i\frac{d}{2b}(k^2-\beta^2)}\cdot e^{-\alpha\pi^2}\int_{-b\pi}^{b\pi} e^{i(\beta-k)t/b}dt\right| \\ &< \left|\sqrt{\frac{-i}{2b\pi}}e^{i\frac{d}{2b}(k^2-\beta^2)}\right|\cdot 2b\pi e^{-\alpha b^2\pi^2}(\frac{1}{b^2\pi^2}+1) \\ &= \sqrt{2b\pi}e^{-\alpha b^2\pi^2}(1+\frac{1}{b^2\pi^2})\end{aligned} \tag{18}$$

$$\left|\sigma_{A,k}-\rho_{A,k}-\sqrt{\frac{-i}{2b\pi}}e^{i\frac{d}{2b}(k^2-\beta^2)}\cdot e^{-\alpha\pi^2}\int_{-b\pi}^{b\pi} e^{i(\beta-k)t/b}dt\right| < \frac{4b^2\alpha\sqrt{2b\pi}e^{-\alpha b^2\pi^2}}{(\beta-k)^2}(1+\frac{1}{b^2\pi^2}) \tag{19}$$

for any $t\in(-b\pi,b\pi)$, combination of Eqs.(10), (18), and (19), we have

$$\begin{aligned}&\left|\phi(t)-\sum_{k=-\infty}^{\infty}\rho_{A,k}\sqrt{\frac{-i}{2b\pi}}e^{-i\frac{a}{2b}t^2+i\frac{kt}{b}-i\frac{d}{2b}k^2} -\frac{1}{2b\pi}e^{-\alpha b^2\pi^2}e^{-i\frac{a}{2b}t^2+i\frac{\beta t}{b}-i\frac{d}{2b}\beta^2}\right| \\ &= \left|\sum_{k=-\infty}^{\infty}(\sigma_{A,k}-\rho_{A,k})\sqrt{\frac{i}{2b\pi}}e^{-i\frac{a}{2b}t^2+i\frac{kt}{b}-i\frac{d}{2b}k^2} -\frac{1}{2b\pi}e^{-\alpha b^2\pi^2}e^{-i\frac{a}{2b}t^2+i\frac{\beta t}{b}-i\frac{d}{2b}\beta^2}\right| \\ &< \left|\sum_{k,|\beta-k|\geq\lfloor b\pi\rfloor}\sqrt{\frac{i}{2b\pi}}e^{-i\frac{a}{2b}t^2+i\frac{kt}{b}-i\frac{d}{2b}k^2}\frac{4b^2\alpha\sqrt{2b\pi}e^{-\alpha b^2\pi^2}}{(\beta-k)^2}\times(1+\frac{1}{b^2\pi^2})\right| + \left|\sum_{k,|\beta-k|<\lfloor b\pi\rfloor}\sqrt{\frac{i}{2b\pi}}e^{-i\frac{a}{2b}t^2+i\frac{kt}{b}-i\frac{d}{2b}k^2}\times\sqrt{2b\pi}e^{-\alpha b^2\pi^2}(1+\frac{1}{b^2\pi^2})\right| \\ &< \sum_{k,|\beta-k|\geq\lfloor b\pi\rfloor}\frac{4b^2\alpha e^{-\alpha b^2\pi^2}}{(k-\beta)^2}(1+\frac{1}{b^2\pi^2}) + 2\lfloor b\pi\rfloor e^{-\alpha b^2\pi^2}(1+\frac{1}{b^2\pi^2}) \\ &< 4\alpha b^2 e^{-\alpha b^2\pi^2}(1+\frac{1}{b^2\pi^2})\times 2\sum_{k=\lfloor b\pi\rfloor}^{\infty}\frac{1}{k^2} + 2\lfloor b\pi\rfloor e^{-\alpha b^2\pi^2}(1+\frac{1}{b^2\pi^2})\end{aligned} \tag{20}$$

Some elementary analysis yields

$$\sum_{k=\lfloor b\pi\rfloor}^{\infty}\frac{1}{k^2} < \frac{1}{\lfloor b\pi\rfloor^2}+\int_{\lfloor b\pi\rfloor}^{\infty}\frac{1}{t^2}dt = \frac{1}{\lfloor b\pi\rfloor^2}+\frac{1}{\lfloor b\pi\rfloor} < \frac{1+3b}{9b^2} \tag{21}$$

and substituting Eq.(21) into Eq.(20), we have

$$\begin{aligned}&\left|\phi(t)-\sum_{k=-\infty}^{\infty}\rho_{A,k}\sqrt{\frac{-i}{2\pi}}e^{-i\frac{a}{2b}t^2+i\frac{kt}{b}-i\frac{d}{b}k^2} -\frac{1}{2b\pi}e^{-\alpha b^2\pi^2}e^{-i\frac{a}{2b}t^2+i\beta t-i\frac{bd}{2}\beta^2}\right| \\ &< e^{-\alpha b^2\pi^2}(1+\frac{1}{9b^2})(\alpha+3\alpha b+8b)\end{aligned} \tag{22}$$

Then, we make use of the triangle inequality and Eq.(22) to obtain

$$\begin{aligned}&\left|\phi(t)-\sum_{k=-\infty}^{\infty}\rho_{A,k}\sqrt{\frac{-i}{2b\pi}}e^{-i\frac{a}{2b}t^2+i\frac{kt}{b}-i\frac{d}{2b}k^2}\right| \\ &< \left|\phi(t)-\sum_{k=-\infty}^{\infty}\rho_{A,k}\sqrt{\frac{-i}{2b\pi}}e^{-i\frac{a}{2b}t^2+i\frac{kt}{b}-i\frac{d}{2b}k^2} -\frac{1}{2b\pi}e^{-\alpha b^2\pi^2}e^{-i\frac{a}{2b}t^2+i\beta t/b-i\frac{d}{2b}\beta^2}\right| + \left|\frac{1}{2b\pi}e^{-\alpha\pi^2}e^{-i\frac{a}{2b}t^2+i\beta t-i\frac{bd}{2}\beta^2}\right| \\ &< e^{-\alpha\pi^2}\left[(1+\frac{1}{9b^2})(\alpha+3\alpha b+8b)+\frac{1}{6b}\right]\end{aligned} \tag{23}$$

Thus, the proof of the Theorem 1 is completed. ■

According to Theorem 1, functions $e^{-\alpha t^2}e^{-i\frac{a}{2b}t^2+i\frac{\beta t}{b}-i\frac{d}{2b}\beta^2}$ can be approximated by linear canonical series whose coefficients are given analytically, and the error of the approximation decreases exponentially as α increases.

The coefficients $\rho_{A,k}$ in Eq.(9) have a peak at $k=[\beta]$ ($[\star]$ is the nearest integer to $\star$), and decay exponentially as $k\to\pm\infty$. We keep only the $q+1$ largest coefficients, where the integer q is chosen such as

$$q \geq 4\alpha b^2\pi \tag{24}$$

Thus we obtain

$$e^{(-q/2)^2/4b^2\alpha} \leq e^{-\alpha b^2\pi^2} \tag{25}$$

The following theorem provides a method for approximating functions $\phi(t)$ defined by a $q+1$ term series, and presents the truncation error under the conditions Eq.(24).

Theorem 2: Let q be an even integer and $q\geq 4\alpha b^2\pi$, a,b,c,d, $\alpha>1/2$ and β be any real. Then, for any $t\in(-b\pi,b\pi)$, the following inequality is obtained

$$\begin{aligned}&\left|\phi(t)-\sum_{k=[\beta]-q/2}^{[\beta]+q/2}\rho_{A,k}\sqrt{\frac{i}{2b\pi}}e^{-i\frac{a}{2b}t^2-i\frac{kt}{b}-i\frac{d}{2b}k^2}\right| \\ &< e^{-\alpha b^2\pi^2}\left[\alpha(1+3b+\frac{1+3b}{9b^2})+8b+\frac{2217}{1052b}\right]\end{aligned} \tag{26}$$

where $\phi(t)=e^{-\alpha t^2}e^{-i\frac{a}{2b}t^2+i\beta t-i\frac{d}{2b}(\beta b)^2}$,$A=[a,b;c,d]$; $\rho_{A,k}$ are defined by Eq.(9).

Proof: For any $t \in (-b\pi, b\pi)$,

$$\begin{aligned}
&\left|\phi(t) - \sum_{k=[\beta]-q/2}^{[\beta]+q/2} \rho_{A,k}\sqrt{\frac{i}{2b\pi}} e^{-i\frac{a}{2b}t^2+i\frac{kt}{b}-i\frac{d}{2b}k^2}\right| \\
&< \left|\phi(t) - \sum_{k=-\infty}^{\infty} \rho_{A,k}\sqrt{\frac{i}{2b\pi}} e^{-i\frac{a}{2b}t^2+i\frac{kt}{b}-i\frac{d}{2b}k^2}\right| \\
&+ \left|\sum_{k<[\beta]-q/2} \rho_{A,k}\sqrt{\frac{i}{2b\pi}} e^{-i\frac{a}{2b}t^2+i\frac{kt}{b}-i\frac{d}{b}k^2}\right| \\
&+ \left|\sum_{k>[\beta]+q/2} \rho_{A,k}\sqrt{\frac{i}{2b\pi}} e^{-i\frac{a}{2b}t^2+i\frac{kt}{b}-i\frac{d}{2b}k^2}\right|
\end{aligned} \tag{27}$$

Due to Eq.(9) and the triangle inequality, we have the inequalities

$$\begin{aligned}
&\left|\sum_{k>[\beta]+q/2} \rho_{A,k}\sqrt{\frac{i}{2b\pi}} e^{-i\frac{a}{2b}t^2+i\frac{kt}{b}-i\frac{d}{2b}k^2}\right| \\
&< \sqrt{\frac{1}{2b\pi}} \sum_{k=[\beta]+q/2+1}^{\infty} \frac{1}{\sqrt{2b\pi}} \\
&\times \left|e^{i\frac{d}{2b}(k^2-\beta^2)}\sqrt{\frac{\pi}{\alpha}} e^{-\frac{(\beta-k)^2}{4b^2\alpha}} e^{-i\frac{a}{2b}t^2+i\frac{kt}{b}-i\frac{d}{2b}k^2}\right| \\
&< \frac{1}{2b\sqrt{\alpha\pi}} \sum_{k=[\beta]+q/2+1}^{\infty} e^{-\frac{(\beta-k)^2}{4b^2\alpha}} \\
&< \frac{1}{2b\sqrt{\alpha\pi}} \sum_{k=q/2}^{\infty} e^{-\frac{k^2}{4b^2\alpha}} < \frac{1}{b\sqrt{2\pi}} \sum_{k=q/2}^{\infty} e^{-\frac{k^2}{4b^2\alpha}}
\end{aligned} \tag{28}$$

$$\begin{aligned}
&\left|\sum_{k<[\beta]-q/2} \rho_{A,k}\sqrt{\frac{i}{2b\pi}} e^{-i\frac{a}{2b}t^2+i\frac{kt}{b}-i\frac{d}{2b}k^2}\right| \\
&< \sqrt{\frac{1}{2b\pi}} \sum_{k=-\infty}^{[\beta]-q/2+1} \frac{1}{\sqrt{2b\pi}} \\
&\times \left|e^{i\frac{d}{2b}(k^2-\beta^2)}\sqrt{\frac{\pi}{\alpha}} e^{-\frac{(\beta-k)^2}{4b^2\alpha}} e^{-i\frac{a}{2b}t^2+i\frac{kt}{b}-i\frac{d}{2b}k^2}\right| \\
&= \frac{1}{2b\sqrt{\alpha\pi}} \sum_{k=[\beta]-q/2+1}^{\infty} e^{-\frac{(\beta-k)^2}{4b^2\alpha}} \\
&< \frac{1}{2b\sqrt{\alpha\pi}} \sum_{k=q/2}^{\infty} e^{-\frac{k^2}{4b^2\alpha}} < \frac{1}{b\sqrt{2\pi}} \sum_{k=q/2}^{\infty} e^{-\frac{k^2}{4b^2\alpha}}
\end{aligned} \tag{29}$$

Because of any real $\alpha > 0$ and $r > 0$, we have

$$\int_r^{\infty} e^{-\alpha t^2} dt < \frac{e^{-\alpha r^2}}{2\alpha r} \tag{30}$$

Thus,

$$\begin{aligned}
&\sum_{k=q/2}^{\infty} e^{-k^2/4b^2\alpha} < e^{-(q/2)^2/4b^2\alpha} + \int_{q/2}^{\infty} e^{-t^2/4b^2\alpha} dt \\
&< e^{-\alpha b^2\pi^2}(1 + \frac{1}{\pi})
\end{aligned} \tag{31}$$

Substituting Eq.(31) into Eqs.(28) and (29), we have

$$\begin{aligned}
&\left|\sum_{k>[\beta]+q/2} \rho_{A,k}\sqrt{\frac{i}{2\pi}} e^{-i\frac{a}{2b}t^2+i\frac{kt}{b}-i\frac{d}{2b}k^2}\right| \\
&+ \left|\sum_{k<[\beta]-q/2} \rho_{A,k}\sqrt{\frac{i}{2\pi}} e^{-i\frac{a}{2b}t^2+i\frac{kt}{b}-i\frac{d}{2b}k^2}\right| \\
&< \frac{2}{b\sqrt{2\pi}} e^{-\alpha b^2\pi^2}(1 + \frac{1}{\pi})
\end{aligned} \tag{32}$$

Substituting Eqs.(23), (32) into Eq.(27), we obtain

$$\begin{aligned}
&\left|\phi(t) - \sum_{k=[\beta]-q/2}^{[\beta]+q/2} \rho_{A,k}\sqrt{\frac{i}{2\pi}} e^{-i\frac{a}{2b}t^2-i\frac{kt}{b}-i\frac{d}{2b}k^2}\right| \\
&< e^{-\alpha b^2\pi^2}\Big[(1 + \frac{1}{9b^2})(\alpha + 3\alpha b + 8b) \\
&+ \frac{1}{6b} + \frac{2}{b\sqrt{2\pi}}(1 + \frac{1}{\pi})\Big] \\
&= e^{-\alpha b^2\pi^2}\left[\alpha(1 + 3b + \frac{1+3b}{9b^2}) + 8b + \frac{2217}{1052b}\right]
\end{aligned} \tag{33}$$

■

Finally, Theorem 3 makes use of a simple linear scaling to generalize the inequality Eq.(26) from $[-b\pi/m, b\pi/m]$ to $[-s, s]$,

Theorem 3: Let $\alpha > 1/2$, β, $s > 0$ be real numbers and $m \geq 2$, $q \geq 4\alpha\pi$ be integers. Then, for any $t \in [-s, s]$

$$\begin{aligned}
&\Big|e^{-i\frac{a}{2b}t^2+i\frac{\beta t}{b}-i\frac{d}{2b}\beta^2} - e^{\alpha(\frac{b\pi t}{ms})^2} \\
&\times \sum_{k=[ms\beta/b\pi]-q/2}^{[ms\beta/b\pi]+q/2} \rho'_{A,k}\sqrt{\frac{i}{2b\pi}} e^{-i\frac{a}{2b}t^2+ik\frac{\pi}{ms}t-i\frac{dk^2}{2b}(\frac{b\pi}{ms})^2}\Big| \\
&< \epsilon
\end{aligned} \tag{34}$$

where

$$\epsilon = e^{-\alpha b^2\pi^2(1-1/m^2)}\left[\alpha(1 + 3b + \frac{1+3b}{9b^2}) + 8b + \frac{2217}{1052b}\right] \tag{35}$$

$$\rho'_{A,k} = \sqrt{\frac{-i}{2b\alpha\pi}} e^{i\frac{d}{2b}(\frac{b\pi}{ms})^2(k^2-(ms\beta/b\pi)^2)} e^{-\frac{(ms\beta/b\pi-k)^2}{4b^2\alpha}} \tag{36}$$

For an integer m and a set of real of numbers t_j, we will denote by v_j the nearest integer to $t_j mN/(2\pi)$ for $j = 0, 1, \cdots, N-1$, and $Q_{j,k}$ a set of real numbers defined by the formula

$$Q_{j,k} = \frac{1}{2\pi\sqrt{\alpha}} e^{-[t_j mN/2\pi-(v_j+k)]^2/4\alpha} \tag{37}$$

Corollary 1: If $a = d = 0, b = 1, c = -1$,$s = N/2$ in Theorem 3, and the conditions of Theorem 3 are satisfied, then we obtain

$$\left|e^{it_j n} - e^{\alpha(\frac{2\pi n}{mN})^2} \sum_{k=-q/2}^{q/2} Q_{j,k} e^{in(k+v_j)2\pi/Nm}\right| < \varepsilon \tag{38}$$

for $j = 0, 1, \cdots, N-1$ and $n \in [-N/2, N/2]$,where ε is defined by

$$\varepsilon = e^{-\alpha\pi^2(1-\frac{1}{m^2})}(5\alpha + 28/3) \tag{39}$$

For a given set of complex numbers $\{\beta_n\}$, we will denote by $\{\beta'_n\}$ is defined as following

$$\beta'_n = \beta_n e^{-i\frac{d}{2b}n^2 + \alpha(2\pi n/mN)^2} \tag{40}$$

for $n = -N/2, \cdots, N/2-1$, and by U_l a set of complex numbers defined by the formula

$$U_l = \sum_{n=-N/2}^{N/2-1} \beta'_n e^{i2\pi nl/mN} \tag{41}$$

for $l = -mN/2, \cdots, mN/2$. Furthermore, taking account of the periodicity of l, we will denote by $\{\widetilde{g_j}\}$ another set of complex numbers defined by the formula

$$\widetilde{g_j} = e^{-i\frac{a}{2b}(bt_j)^2} \sum_{l=-q/2}^{q/2} Q_{j,l} U_{v_j+l} \tag{42}$$

for $j = 0, 1, \cdots, N-1$.

Combining Eqs.(38)-(42) with the triangle inequality, we see that

$$|g_j - \widetilde{g_j}| < \varepsilon \sum_{n=0}^{N-1} |\beta_n| \tag{43}$$

for $j = 0, 1, \cdots, N-1$, where $\{g_j\}$ are defined by Eq.(6). Thus, the implements of NFLCT is given in Table I.

In the following section, we will give a numerical example to support our theoretical analysis in the above section which show the high efficiency of NFLCT in this paper.

IV. Simulations

In this part, two measures of precision are selected for the NFLCT algorithm.

$$E_\infty = \max_{0\le j\le N-1} \left|\tilde{f}_j - f_j\right| / \sum_{j=0}^{N-1} |\alpha_j| \tag{44}$$

$$E_2 = \sqrt{\sum_{j=0}^{N-1} \left|\tilde{f}_j - f_j\right|^2 / \sum_{j=0}^{N-1} |f_j|^2} \tag{45}$$

where α is the input data, f is the result of a direct computation, and $\tilde{f}_j$ is the result of computation by proposed methods.

Example Here we consider the transformation $G : C^N \to C^N$ as defined by the formula

$$g_j = G(\mathbf{g})_j = \sum_{k=-N/2}^{N/2-1} \beta_k e^{-i\frac{a}{2b}t_j^2 + i\frac{kt_j}{b} - i\frac{d}{2b}k^2} \tag{46}$$

for $j = 0, \cdots, N-1$. In this example, $t_0, \cdots, t_{N-1}$ were randomly distributed on the interval $[-b\pi, b\pi]$, and $\beta_k = e^{-i2k^2+i3m_k}$, $k = -N/2, \cdots, N/2-1$, m_k were distributed randomly on the interval $[-N/2, N/2-1]$.

We take the parameters $a = 2, b = 1, c = 7, d = 4$, the interpolate factors $m = 2$, the terms of LCS $q = 10$, $\alpha = 0.646$. For $N = 64$, we loop our algorithm 20 times, the results are presented in Figure 1. The error of amplitude between the directly NDLCT and the proposed algorithm plots in Figure 2. It shows that the proposed algorithm has almost the same effective as the direct method. For different N, the results of E_∞ and E_2 are showed in Table II. It suggested that the precision of Algorithm2 is independent N.

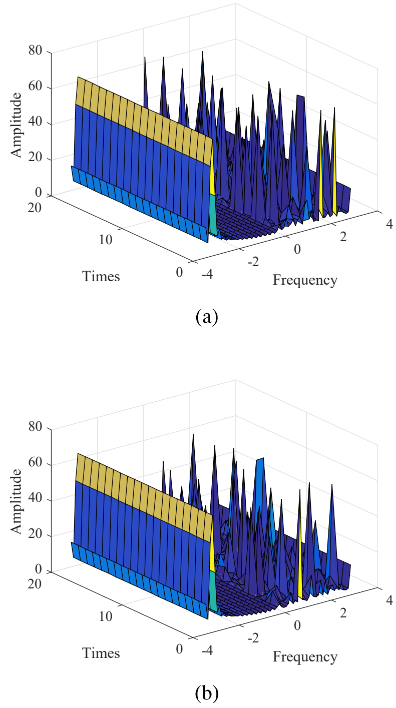

Fig. 1. Twenty independent experiments. Amplitude (a) obtained by directly summation Eq.46; (b) obtained by NFLCT.

V. Conclusions

In this paper, we have described a algorithm for computing DLCT for nonequispaced data, which based on the interpolation formulae to transform function values from equispace to nonequispaced points. The simulation shows that the derived approach is effevtive for computing nonuniform DLCT. The proposed algorithm can be viewed as generalizations of discrete linear canonical transform, and will have a broad range of applications in many branches of mathematics, science and engineering.

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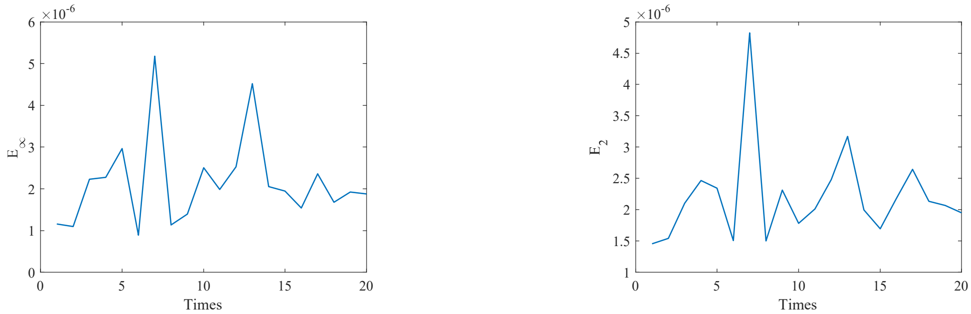
TABLE I
ALGORITHM FOR FAST COMPUTATION OF EQ.(6)

Algorithm 2 NFLCT for fast computation of Eq.(6)

Input parameter $A = [a, b; c, d]$, the $\{-N/2, \cdots, N/2 - 1\}$, $\{\beta_{-N/2}, \cdots, \beta_{N/2-1}\}$, $\{t_0, \cdots, t_{N-1}\}$.
Choose precision ϵ α and $q = \lceil 4\pi b^2 \alpha \rceil$
for $j = 0 : N - 1$
Compute v_j, the nearest integer to $t_j mN/(2\pi)$
Calculate $\beta'_{j-N/2} = \beta_{j-N/2} e^{-id/2b(j-N/2)^2 + \alpha[(2\pi(j-N/2))/(mN)]^2}$ according to Eq.(40)
for $k = -q/2 : q/2$
Calculate $Q_{j,k}$ according to Eq.(37).
end
end
Comment: Evaluate Fourier series at uniform sampling in $[-bm\pi, bm\pi]$ using inverse FFT of size mN
Calculate $U_l = \sum_{n=-N/2}^{N/2} \beta'_n e^{i2\pi nl/mN}$, for $l = -mN/2, \cdots, mN/2 - 1$
Comment: Calculate approximate values at desired points in terms of the values at equispaced points $[-b\pi, b\pi]$
for $j = 0 : N - 1$
for $l = -q/2 : q/2$
$\widetilde{g_j} = \widetilde{g_j} + Q_{j,l} U_{v_j+l}$
end
Output :Approximate values $\widetilde{g_j} = e^{-ia/(2b)(bt_j)^2} \widetilde{g_j}, j = -N/2, \cdots, N/2 - 1$.
The total complexity is $Nq + mN/2 \log_2 N + 3N$



(a) (b)

Fig. 2. Twenty independent experiments. Error of amplitude between the directly NDLCT and NFLCT (a) E_∞; (b) E_2

TABLE II
ERROR COMPARISON IN CASE OF DIFFERENCE N .

N	64	128	256	512	1024
E_∞	2.1569×10^{-6}	2.0019×10^{-6}	2.1367×10^{-6}	2.0761×10^{-6}	2.0611×10^{-6}
E_2	2.1113×10^{-6}	2.2353×10^{-6}	2.2271×10^{-6}	2.0740×10^{-6}	2.4019×10^{-6}

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