Constrained Design of Two-Dimensional FIR Filters with Sparse Coefficients

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***Abstract*—We present an algorithm for the constrained design of a 2D FIR filter with sparse coefficients. Existing filter design methods aim to minimize a filter order and maximize filter performance. The 2D FIR filter coefficients designed by the least-squares method with peak error constraints are optimal in the sense of least-squares within a given order. However, they are not necessarily optimal in terms of constructing a filter that satisfies the design specification. That is, a higher-order filter with some zero coefficients can construct a filter that satisfies the design specification with fewer multipliers. Our method minimizes the number of non-zero coefficients of the filter coefficients, while the frequency response of the filter satisfies the design specification. It performs better in terms of maximum error than the least-squares method with peak error constraints having the same number of multipliers.**

***Index Terms*—2D FIR filter, sparsity, ℓ_0 norm, ADMM algorithm.**

I. Introduction

The design of FIR filters is an important issue in digital signal processing. Many design methods have been proposed by several authors [1]–[12]. Existing filter design methods aim to minimize a filter order and maximize the filter performance. However, as the filter length increases, the number of multipliers used to construct the filter increases. It is a serious problem, especially in a two-dimensional (2D) FIR filter design.

The *least-squares* (LS) method, which minimizes the mean squared error of frequency responses, is widely used due to its simplicity and flexibility [3]–[7]. In the LS design, large error often occurs near a cut-off frequency. Adam et al. have addressed the problem by adding constraints to the minimization problem of the filter design [13], which is termed as the *constrained least-squares* (CLS) method. In this method, the maximum error is reduced by adding the peak error constraints to its frequency response without having large transition bands. Under an arbitrary maximum error constraint, the filter coefficients designed by the CLS method are optimal in the sense of LS at a given filter length, but not necessarily optimal among filters with the same number of multipliers in terms of constructing a filter that meets the design specification. If we are allowed to have longer filter lengths, we can design a filter that satisfies the condition with fewer multipliers than the CLS method. To minimize the number of multipliers instead of the filter order, some approaches design the filter with zero-valued taps, which is often called *sparse filters* [14], [15].

In this paper, we present a numerical approach to design a constrained 2D sparse filter. Our method consists of two steps. In the first step, we obtain an approximate solution to the peak error-constrained L_0 norm minimization problem, which identifies the positions of the zero coefficients. In the second step, a constrained sparse filter is designed by solving a constrained filter design problem that includes arbitrary constraints for the frequency response and filter coefficients. Although this method does not guarantee optimality in the sense of sparsity, it has better performance than the traditional 2D filter design method.

The paper is organized as follows. In Section II, the weighted LS and CLS for the FIR filters are briefly described. In Section III, our design problem is formulated and a design algorithm that considers the sparsity of coefficients and the arbitrary constraint for peak errors is proposed. In Section IV, several examples are shown to verify the validity of the proposed algorithm, and some comparisons with the conventional method are shown. In the last Section V, we briefly conclude this paper.

II. Conventional Methods

A. Weighted least-squares method

The frequency response of 2D linear phase FIR filters and more general 2D non-linear phase FIR filters can be written as a linear combination of trigonometric basis functions. If the filter is a 2D symmetric linear phase filter of even-order even-symmetry, it can be written as follows,

$$H(\omega_1, \omega_2) = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} A_{n,m} \phi_{n,m}(\omega_1, \omega_2), \tag{1}$$
$$\phi_{n,m}(\omega_1,\ \omega_2) = \cos(n\omega_1)\cos(m\omega_2),$$
$$\{(\omega_1, \omega_2),\ 0 \le \omega_1, \omega_2 \le \pi\},$$

where $N = (N_0 - 1)/2 + 1$, N_0 is the filter length and $A_{n,m}$ is the (n, m)-th elements of the filter coefficient matrix $\mathbf{A} \in \mathbb{R}^{N \times N}$. In the more general case, $\phi_{n,m}(\omega_1, \omega_2) = e^{jn\omega_1} e^{jm\omega_2}$. Although our design method can handle all types of 2D FIR filters, we only show here the case of 2D linear phase FIR filters whose frequency response is indicated by (1). Here, we briefly review the *weighted least-squares* (WLS) method. In general, the WLS method is defined as the problem of minimizing the following cost function, expressed as a finite sum of the errors on the discretized frequency points as follows,

$$\Phi = \sum_{i=0}^{K-1}\sum_{j=0}^{K-1} |W(\omega_{1,i},\omega_{2,j})|^2 |H(\omega_{1,i},\omega_{2,j}) - D(\omega_{1,i},\omega_{2,j})|^2, \quad \{(\omega_{1,i},\omega_{2,i}), i=0,1,\ldots,K-1\}, \tag{2}$$

where $W(\omega_{1,i},\omega_{2,j})$ is a weight function with zero or positive values, $D(\omega_{1,i},\omega_{2,j})$ is the desired frequency response, and K is the number of frequency response samples. $\omega_{1,i}$ and $\omega_{2,i}$ are the i-th frequency of the frequency sample along the ω_1 and ω_2 axes, respectively, which is $\{(\omega_{1,i},\omega_{2,i}), i = 0,1,\ldots,K-1\} \subseteq \{(\omega_1,\omega_2),\ 0 \le \omega_1,\omega_2 \le \pi\}$. The filter coefficient vector obtained by vectorizing the filter coefficient matrix $\mathbf{A}$ is defined as $\mathbf{a}$, and then we denote it by using the L_2 norm of the error.

$$\Phi^{1/2} = \|\mathbf{W}(\mathbf{Ra}-\mathbf{d})\|_2. \tag{3}$$

Here, the (i,j)-th element of $\mathbf{R}$, the i-th element of $\mathbf{d}$, and the (i,i)-th element of $\mathbf{W}$ are given by

$$\begin{aligned}
\mathbf{R}_{i,j} &= R_{\lfloor i/N\rfloor,(i \bmod N)}(\omega_{1,\lfloor j/K\rfloor},\omega_{2,(j \bmod K)}),\\
R_{n,m}(\omega_i,\omega_j) &= \cos(n\omega_i)\cos(m\omega_j),\\
\mathbf{d}_i &= D(\omega_{1,\lfloor i/K\rfloor},\omega_{2,(i \bmod K)}),\\
\mathbf{W}_{i,i} &= W(\omega_{1,\lfloor i/K\rfloor},\omega_{2,(i \bmod K)}),\\
&\{(\omega_{1,i},\omega_{2,i}), i=0,1,\ldots,K-1\},
\end{aligned}$$

which are $\mathbf{R} \in \mathbb{R}^{K^2\times N^2}, \mathbf{d} \in \mathbb{R}^{K^2}, \mathbf{W} \in \mathbb{R}^{K^2\times K^2}$, and $\mathbf{a} \in \mathbb{R}^{N^2}$. $\mathbf{W}$ is a diagonal matrix. The optimal filter coefficients $\mathbf{a}$ in the sense of LS can be uniquely determined by solving the normal equation.

$$\mathbf{a} = (\mathbf{R}^{\mathrm{T}}\mathbf{W}^2\mathbf{R})^{-1}\mathbf{R}^{\mathrm{T}}\mathbf{W}^2\mathbf{d}.$$

This is the general WLS method for 2D linear phase FIR filters.

B. Peak Error Constraint

In the filter design, the LS approximation with peak error constraints is useful in several applications. The L_2 problem with peak error constraints for a 2D low-pass filter with passband/stopband edge ω_p,ω_s is defined as

$$\begin{aligned}
\min_{\mathbf{a}} \sum_{i=0}^{K-1}\sum_{j=0}^{K-1} & |W(\omega_{1,i},\omega_{2,j})|^2 |H(\omega_{1,i},\omega_{2,j}) - D(\omega_{1,i},\omega_{2,j})|^2,\\
s.t.\ & L(\omega_{1,i},\omega_{2,j}) \le H(\omega_{1,i},\omega_{2,j}) \le U(\omega_{1,i},\omega_{2,j}),\\
& \{(\omega_{1,i},\omega_{2,i}), i=0,1,\ldots,K-1\}.
\end{aligned} \tag{4}$$

Here, $L(\omega_1,\omega_2)$ and $U(\omega_1,\omega_2)$ are the specified lower and upper bound functions, respectively. These functions are given by

$$L(\omega_1,\omega_2) = \begin{cases} 1-\delta_p, & if\ \omega_1,\omega_2 \in \Omega_p,\\ -\delta_s, & if\ \omega_1,\omega_2 \in \Omega_s,\\ & don't\ care\ otherwise, \end{cases} \tag{5}$$

$$U(\omega_1,\omega_2) = \begin{cases} 1+\delta_p, & if\ \omega_1,\omega_2 \in \Omega_p,\\ \delta_s, & if\ \omega_1,\omega_2 \in \Omega_s,\\ & don't\ care\ otherwise, \end{cases} \tag{6}$$

where δ_p and δ_s are the given error bounds in the passband and stopband, and Ω_p and Ω_s are the set of frequency response sample points in the passband and stopband, respectively. The filter with peak error constraints is designed by solving (4).

III. Proposed Method

The WLS method is optimal in the sense of LS, under the condition that the filter length is fixed. If we set some of the coefficients of the filter coefficient vector to zero and allow a longer filter length, we can design the filter such that it satisfies the design specification with fewer multipliers. Our goal is to design a 2D sparse filter with such constraints. The proposed design algorithm consists of the following two steps. We first determine the position where the filter coefficients are set to zero by solving a peak error-constrained L_0 norm minimization problem (described in Section III-A). We then solve an arbitrary constrained sparse filter design problem (described in Section III-B) to bring the coefficients determined in the first step to zero.

A. Determination of zero coefficients (step1)

To design sparse coefficients, we use the L_0 norm. For the input coefficient vector $\mathbf{x}$, the L_0 norm is defined by

$$\|\mathbf{x}\|_0 = \sum_i |x_i|^0, \tag{7}$$

where we define

$$|x_i|^0 = \begin{cases} 1, & if\ x_i \ne 0,\\ 0, & if\ x_i = 0. \end{cases} \tag{8}$$

We define a sparse filter design problem such that the function (7) is minimized while satisfying the peak error constraints.

$$\begin{aligned}
&\min_{\mathbf{a}}\ \lambda\|\mathbf{a}\|_0,\\
&s.t.\ \hat{\mathbf{W}}\mathbf{Ra} \in V_1,\\
&V_1 = \{\mathbf{x}|\mathbf{L}_{\lfloor i/K\rfloor,(i \bmod K)} \le x_i \le \mathbf{U}_{\lfloor i/K\rfloor,(i \bmod K)}\},
\end{aligned} \tag{9}$$

where

$$\mathbf{L}_{i,j} = \begin{cases} L(\omega_i,\omega_j), & if\ \omega_i,\omega_j \in \Omega_p \cup \Omega_s,\\ 0, & otherwise, \end{cases} \tag{10}$$

$$\mathbf{U}_{i,j} = \begin{cases} U(\omega_i,\omega_j), & if\ \omega_i,\omega_j \in \Omega_p \cup \Omega_s,\\ 0, & otherwise, \end{cases} \tag{11}$$

$$\hat{\mathbf{W}}_{i,i} = \begin{cases} 1, & if\ \omega_{\lfloor i/K\rfloor},\omega_{(i \bmod K)} \in \Omega_p \cup \Omega_s,\\ 0, & otherwise, \end{cases} \tag{12}$$

Algorithm 1 Algorithm1

input $\eta, \beta_1, \beta_2, k = 0, \mathbf{a}^{(0)}, \mathbf{z}_n^{(0)}, \mathbf{b}_n^{(0)}, (n = 1, 2)$

while *A stopping criterion is not satisfied* **do**

$$\mathbf{a}^{(k+1)} \leftarrow \arg\min_{\mathbf{a}} \frac{\beta_1}{2}\|\mathbf{z}_1^{(k)} - \mathbf{a} - \mathbf{b}_1^{(k)}\|_2^2 + \frac{\beta_2}{2}\|\mathbf{z}_2^{(k)} - \hat{\mathbf{W}}\mathbf{R}\mathbf{a} - \mathbf{b}_2^{(k)}\|_2^2 \tag{13}$$

$$\mathbf{z}_1^{(k+1)} \leftarrow \arg\min_{\mathbf{z}_1} \lambda\|\mathbf{z}_1\|_0 + \frac{\beta_1}{2}\|\mathbf{z}_1 - \mathbf{a}^{(k+1)} - \mathbf{b}_1^{(k)}\|_2^2 \tag{14}$$

$$\mathbf{z}_2^{(k+1)} \leftarrow \arg\min_{\mathbf{z}_2} \iota_{V_1}(\mathbf{z}_2) + \frac{\beta_2}{2}\|\mathbf{z}_2 - \hat{\mathbf{W}}\mathbf{R}\mathbf{a}^{(k+1)} - \mathbf{b}_2^{(k)}\|_2^2 \tag{15}$$

$\mathbf{b}_1^{(k+1)} \leftarrow \mathbf{b}_1^{(k)} + \mathbf{a}^{(k+1)} - \mathbf{z}_1^{(k+1)}$

$\mathbf{b}_2^{(k+1)} \leftarrow \mathbf{b}_2^{(k)} + \hat{\mathbf{W}}\mathbf{R}\mathbf{a}^{(k+1)} - \mathbf{z}_2^{(k+1)}$

$\beta_1 \leftarrow \eta\beta_1$

$\beta_2 \leftarrow \eta\beta_2$

end while

Output : $\mathbf{a}$

and $\mathbf{L}, \mathbf{U} \in \mathbb{R}^{K\times K}$, $\hat{\mathbf{W}} \in \mathbb{R}^{K^2\times K^2}$. L and U are the functions defined in (5) and (6), respectively. $\hat{\mathbf{W}}$ is a diagonal matrix, which extracts the frequency responses belonging to the passband and stopband of the constructed filter. where $\mathbf{R}$ is the basis function matrix introduced in Section II-A. The parameter λ is introduced to control the sparsity of the filter coefficient vector. $\hat{\mathbf{W}}\mathbf{R}\mathbf{a} \in V_1$ is a condition for the frequency response of the passband and stopband to satisfy the peak error constraints. Here, we rewrite the sparse filter design problem (9) to an unconstrained problem by using the *indicator function*.

$$\min_{\mathbf{a}} \lambda\|\mathbf{a}\|_0 + \iota_{V_1}(\hat{\mathbf{W}}\mathbf{R}\mathbf{a}), \tag{16}$$
$$V_1 = \{\mathbf{x}|\mathbf{L}_{\lfloor i/K\rfloor,(i \bmod K)} \le x_i \le \mathbf{U}_{\lfloor i/K\rfloor,(i \bmod K)}\},$$

where ι_{V_1} is the indicator function of V_1. Here, $\iota_V(\mathbf{x})$ penalizes a vector $\mathbf{x}$ if it is not included in the set V.

$$\iota_V(\mathbf{x}) = \begin{cases} 0, & if\ \mathbf{x} \in V, \\ +\infty, & if\ \mathbf{x} \notin V. \end{cases} \tag{17}$$

An approximate solution is obtained by applying the *Alternating Direction Method of Multipliers*(ADMM) [16]. ADMM is a proximal splitting algorithm that can treat convex optimization problems. Although ADMM is used as an algorithm to solve convex optimization problems, it has been successfully applied to various applications in non-convex optimization problems [17], [18]. Therefore, ADMM is used for the non-convex optimization problems of (16). The algorithm for solving (16) is shown in Algorithm 1.

Here, $\eta \in \mathbb{R}$ is the step size to update β_1, β_2 and has a value greater than 1. The reasonable solution is obtained by solving the problem repeatedly while varying β_1 and β_2 by multiplying η in each step. For the initial values of β_1 and β_2, a constant greater than zero is set, respectively.

1) solution of $\mathbf{a}$*:* The problem (13) is a simple quadratic form w.r.t. $\mathbf{a}$. The optimal solution is determined by solving

$$(\beta_1\mathbf{I} + \beta_2\mathbf{R}^{\mathrm{T}}\hat{\mathbf{W}}^2\mathbf{R})\mathbf{a} = \beta_1(\mathbf{z}_1 - \mathbf{b}_1) + \beta_2\mathbf{R}^{\mathrm{T}}\hat{\mathbf{W}}^{\mathrm{T}}(\mathbf{z}_2 - \mathbf{b}_2). \tag{18}$$

Here, $\mathbf{I} \in \mathbb{R}^{N^2\times N^2}$ is an identity matrix.

2) solution of $\mathbf{z}_1$*:* Reformulating (14), we get

$$\mathbf{z}_1^{(k+1)} = \arg\min_{\mathbf{z}_1} \|\mathbf{z}_1\|_0 + \frac{\beta_1}{2\lambda}\|\mathbf{z}_1 - \mathbf{a}^{(k+1)} - \mathbf{b}_1^{(k)}\|_2^2.$$

The solution of $\mathbf{z}_1$ is found for each element individually.

$$z_{1,i}^{(k+1)} = \arg\min_{z_{1,i}} |z_{1,i}|^0 + \frac{\beta_1}{2\lambda}(z_{1,i} - a_i^{(k+1)} - b_{1,i}^{(k)})^2, \tag{19}$$

where a_i, $z_{1,i}$, and $b_{1,i}$ are the i-th element of $\mathbf{a}$, $\mathbf{z}_1$, and $\mathbf{b}_1$ respectively. The solution $z_{1,i}^*$ can be obtained by hard thresholding given by (19) as follows,

$$z_{1,i}^* = \begin{cases} 0, & if\ |a_i + b_{1,i}| \le \sqrt{\frac{2\lambda}{\beta_1}}, \\ a_i + b_{1,i}, & otherwise. \end{cases} \tag{20}$$

3) solution of $\mathbf{z}_2$*:* The solution of $\mathbf{z}_2$ from (15) is found separately for each element as follows,

$$z_{2,i}^* = \arg\min_{z_{2,i}} \iota_{V_1}(z_{2,i}) + \frac{\beta_2}{2}(z_{2,i} - (\hat{\mathbf{W}}\mathbf{R}\mathbf{a})_i - b_{2,i})^2, \tag{21}$$

where $(\hat{\mathbf{W}}\mathbf{R}\mathbf{a})_i, z_{2,i}$, and $b_{2,i}$ are the i-th element of $\hat{\mathbf{W}}\mathbf{R}\mathbf{a}, \mathbf{z}_2$, and $\mathbf{b}_2$ respectively. One can easily find that the solution for the above problem is given by the projection onto the convex set V_1 as

$$z_{2,i}^* = \begin{cases} \mathbf{U}_{x,y}, & if\ (\hat{\mathbf{W}}\mathbf{R}\mathbf{a})_i + b_{2,i} > \mathbf{U}_{x,y}, \\ \mathbf{L}_{x,y}, & if\ (\hat{\mathbf{W}}\mathbf{R}\mathbf{a})_i + b_{2,i} < \mathbf{L}_{x,y}, \\ (\hat{\mathbf{W}}\mathbf{R}\mathbf{a})_i + b_{2,i}, & otherwise, \end{cases} \tag{22}$$
$$x = \lfloor i/K\rfloor,\ y = (i \bmod K),$$

where $\mathbf{L}$ and $\mathbf{U}$ are the matrices defined in (10) and (11), respectively.

A hard threshold is applied to the obtained solution and the coefficient values of the filter coefficient vectors below the threshold $\xi_1 \in \mathbb{R}$ are set to zero. Let the zero-value index of the filter coefficient vector be the zero coefficient position. This is used in $\mathbf{M}$ of Section III-B.

B. Design of Constrained Sparse Filter (step2)

The purpose of the previous section is to find the positions of the zero coefficients of the filter coefficient vector. Once the position of the zero coefficients is obtained, we redesign

Algorithm 2 Algorithm2

1: **input**

$$k=0, \mathbf{z}_n^{(0)}, \mathbf{b}_n^{(0)}, (n=1,2), \mathbf{a}^{(0)}=\mathbf{a}^*_{\text{step1}}, \mathbf{M}, \mu_1, \mu_2$$

2: **while** *A stopping criterion is not satisfied* **do**

3:
$$\begin{aligned}\mathbf{a}^{(k+1)} \leftarrow \arg\min_{\mathbf{a}} \ &\frac{1}{2}\|\mathbf{W}(\mathbf{Ra}-\mathbf{d})\|_2^2 \\ &+\frac{\mu_1}{2}\|\mathbf{z}_1^{(k)}-\hat{\mathbf{W}}\mathbf{Ra}-\mathbf{b}_1^{(k)}\|_2^2 \\ &+\frac{\mu_2}{2}\|\mathbf{z}_2^{(k)}-\mathbf{Ma}-\mathbf{b}_2^{(k)}\|_2^2\end{aligned} \tag{25}$$

4:
$$\begin{aligned}\mathbf{z}_1^{(k+1)} \leftarrow \arg\min_{\mathbf{z}_1} \ &\iota_{V_1}(\mathbf{z}_1) \\ &+\frac{\mu_1}{2}\|\mathbf{z}_1-\hat{\mathbf{W}}\mathbf{Ra}^{(k+1)}-\mathbf{b}_1^{(k)}\|_2^2\end{aligned} \tag{26}$$

5:
$$\mathbf{z}_2^{(k+1)} \leftarrow \arg\min_{\mathbf{z}_2} \ \iota_{V_2}(\mathbf{z}_2)+\frac{\mu_2}{2}\|\mathbf{z}_2-\mathbf{Ma}^{(k+1)}-\mathbf{b}_2^{(k)}\|_2^2 \tag{27}$$

6: $\mathbf{b}_1^{(k+1)} \leftarrow \mathbf{b}_1^{(k)}+\hat{\mathbf{W}}\mathbf{Ra}^{(k+1)}-\mathbf{z}_1^{(k+1)}$

7: $\mathbf{b}_2^{(k+1)} \leftarrow \mathbf{b}_2^{(k)}+\mathbf{Ma}^{(k+1)}-\mathbf{z}_2^{(k+1)}$

8: **end while**

9: **Output** : $\mathbf{a}$

the filter coefficients to obtain the optimal filter coefficients by solving the following CLS problem with sparsity constraints.

$$\begin{aligned}&\min_{\mathbf{a}} \ \frac{1}{2}\|\mathbf{W}(\mathbf{Ra}-\mathbf{d})\|_2^2, \\ &s.t. \ \hat{\mathbf{W}}\mathbf{Ra}\in V_1, \ \mathbf{Ma}\in V_2,\end{aligned} \tag{23}$$

where $\mathbf{M}$ is the diagonal matrix for extracting the coefficients of zero coefficient position. If the i-th element of $\mathbf{a}$ obtained in Section III-A is 0, the (i,i)-th element of $\mathbf{M}$ is 1; otherwise, it is 0. V_2 represents the constraints for the sparsity of the filter coefficients, $\hat{\mathbf{W}}\mathbf{Ra}$ is a linear constraint added to $\mathbf{a}$. V_1 and V_2 are closed convex sets. We convert (23) to the following unconstrained problem by introducing the indicator function (17),

$$\min_{\mathbf{a}} \ \frac{1}{2}\|\mathbf{W}(\mathbf{Ra}-\mathbf{d})\|_2^2+\iota_{V_1}(\hat{\mathbf{W}}\mathbf{Ra})+\iota_{V_2}(\mathbf{Ma}). \tag{24}$$

The second and third terms of the cost function (24) are non-differentiable functions. Fortunately, since each term of the cost function is convex, the cost function can be solved by ADMM, and its algorithm is shown in Algorithm 2. $\mathbf{a}^*_{\text{step1}}$ is the filter coefficient vector $\mathbf{a}$ after hard thresholding obtained in step 1.

1) solution of $\mathbf{a}$*:* The problem (25) is a simple quadratic form w.r.t. $\mathbf{a}$. The optimal solution is determined by solving

$$\begin{aligned}&(\mathbf{R}^{\mathrm{T}}\mathbf{W}^2\mathbf{R}+\mu_1\mathbf{R}^{\mathrm{T}}\hat{\mathbf{W}}^2\mathbf{R}+\mu_2\mathbf{M}^{\mathrm{T}}\mathbf{M})\mathbf{a}= \\ &\quad \mathbf{R}^{\mathrm{T}}\mathbf{W}^2\mathbf{d}+\mu_1\mathbf{R}^{\mathrm{T}}\hat{\mathbf{W}}^{\mathrm{T}}(\mathbf{z}_1-\mathbf{b}_1)+\mu_2\mathbf{M}^{\mathrm{T}}(\mathbf{z}_2-\mathbf{b}_2).\end{aligned} \tag{28}$$

2) solution of $\mathbf{z}_1$*:* The solution of $\mathbf{z}_1$ is found for each element individually.

$$\begin{aligned}z_{1,i}^{(k+1)} = \arg\min_{z_{1,i}} \ &\iota_{V_1}(z_{1,i}) \\ &+\frac{\mu_1}{2}\|(\hat{\mathbf{W}}\mathbf{Ra}^{(k+1)})_i-z_{1,i}-b_{1,i}^{(k)}\|_2^2,\end{aligned} \tag{29}$$

where $(\hat{\mathbf{W}}\mathbf{Ra})_i$, $z_{1,i}$, and $b_{1,i}$ are the i-th element of $\hat{\mathbf{W}}\mathbf{Ra}$, $\mathbf{z}_1$, and $\mathbf{b}_1$ respectively. One can easily find that the solution for the above problem is given by the projection onto the convex set V_1 as

$$\begin{aligned}&z_{1,i}^* = \begin{cases}\mathbf{U}_{x,y}, & if \ (\hat{\mathbf{W}}\mathbf{Ra})_i+b_{1,i}>\mathbf{U}_{x,y}, \\ \mathbf{L}_{x,y}, & if \ (\hat{\mathbf{W}}\mathbf{Ra})_i+b_{1,i}<\mathbf{L}_{x,y}, \\ (\hat{\mathbf{W}}\mathbf{Ra})_i+b_{1,i}, & otherwise,\end{cases} \\ &x=\lfloor i/K\rfloor, \ y=(i \bmod K).\end{aligned} \tag{30}$$

3) solution of $\mathbf{z}_2$*:* The closed-convex set V_2 is represented by the following equation.

$$V_2=\{\mathbf{x}\in\mathbb{R}^{N^2} \mid |x_i|\le\epsilon \ for \ \forall_i\}.$$

$\iota_{V_2}(\mathbf{x})$ imposes a penalty if all elements of the vector $\mathbf{x}$ are not less than or equal to ϵ. If $\epsilon=0$, we constrain the coefficients value of the zero coefficient position obtained in Section III-A to be zero. We relax this constraint by setting a small value for ϵ. The solution of (27) is given by the projection of the convex set V_2.

$$z_{2,i}^* = \begin{cases}\epsilon, & if \ (\mathbf{Ma})_i+b_{2,i}>\epsilon, \\ -\epsilon, & if \ (\mathbf{Ma})_i+b_{2,i}<-\epsilon, \\ (\mathbf{Ma})_i+b_{2,i}, & otherwise.\end{cases} \tag{31}$$

A hard threshold is applied to the obtained solution and the coefficient values of the filter coefficient vectors below the threshold ξ_2 are set to zero.

IV. Experiment

In this section, numerical experiments are shown to verify the advantage of the proposed algorithm. All experiments were designed in MATLAB. All frequencies are normalized by π and frequency points are equally spaced. The filter lengths N_0 are 61 ($N=31$), 91 ($N=46$) and 141 ($N=71$), and δ_p, δ_s are 0.1, 0.05 and 0.015 for the respective filter lengths. For the cutoff frequency ω_c, the edges of the passband and the stopband ω_p and ω_s are $\omega_c\pm0.02$, respectively. The W used in the experiments is a function that returns 1 when a frequency point belongs to the passband or stopband and 0 otherwise. The parameters used in Algorithm 1 are $\eta=1.001$, $\xi_1=1e-6$, $\beta_1=10$ and $\beta_2=10$, respectively and $\mathbf{a}^{(0)}$ is initialized by $(\mathbf{R}^{\mathrm{T}}\mathbf{W}^2\mathbf{R})^{-1}\mathbf{R}^{\mathrm{T}}\mathbf{W}^2\mathbf{d}$. The parameters used in Algorithm 2 are $\xi_2=1e-6$, $\mu_1=200$ and $\mu_2=1$, respectively, and $\mathbf{a}^{(0)}$ is initialized with the solution obtained in step 1 after hard thresholding.

We designed a low-pass filter with cutoff frequencies ω_c of 0.26, 0.5, and 0.74 and a band-pass filter with passband frequencies $Omega_p$ of $[0.22, 0.38]$, $[0.42, 0.58]$, and

TABLE I: Experimental Results (Low-pass Filter)

$(N^2, N_d, N_{CLS}, \omega_c)$	CLS		Ours	
	$E(U)$	$E(L)$	$E(U)$	$E(L)$
(961, 207, 225, 0.26)	2.01e-1	1.75e-1	1.00e-1	1.00e-1
(961, 250, 256, 0.50)	1.78e-1	1.68e-1	1.00e-1	1.00e-1
(961, 292, 324, 0.74)	1.50e-1	1.37e-1	1.00e-1	1.00e-1
(2116, 1402, 1444, 0.26)	1.55e-1	1.54e-1	5.00e-2	5.00e-2
(2116, 1150, 1156, 0.50)	1.35e-1	1.24e-1	5.00e-2	5.00e-2
(2116, 1072, 1089, 0.74)	1.31e-1	1.17e-1	5.00e-2	5.00e-2
(5041, 3060, 3136, 0.26)	5.72e-2	4.40e-2	1.50e-2	1.50e-2
(5041, 2836, 2916, 0.50)	4.27e-2	4.35e-2	1.50e-2	1.50e-2
(5041, 2608, 2704, 0.74)	3.41e-2	3.20e-2	1.50e-2	1.50e-2

TABLE II: Experimental Results (Band-pass Filter)

$(N^2, N_d, N_{CLS}, passband)$	CLS		Ours	
	$E(U)$	$E(L)$	$E(U)$	$E(L)$
(961, 229, 256, 0.2-0.4)	1.56e-1	1.46e-1	1.00e-1	1.00e-1
(961, 263, 289, 0.4-0.6)	1.49e-1	1.39e-1	1.00e-1	1.00e-1
(961, 545, 576, 0.6-0.8)	1.90e-1	1.85e-1	1.00e-1	1.00e-1
(2116, 732, 784, 0.2-0.4)	6.33e-2	6.45e-2	5.00e-2	5.00e-2
(2116, 707, 729, 0.4-0.6)	8.11e-2	8.19e-2	5.00e-2	5.00e-2
(2116, 716, 729, 0.6-0.8)	6.90e-2	6.58e-2	5.00e-2	5.00e-2
(5041, 1878, 1936, 0.2-0.4)	2.16e-2	2.12e-2	1.50e-2	1.50e-2
(5041, 2026, 2116, 0.4-0.6)	1.68e-2	1.67e-2	1.50e-2	1.50e-2
(5041, 2128, 2209, 0.6-0.8)	1.54e-2	1.53e-2	1.50e-2	1.50e-2

$[0.62, 0.78]$. The band-pass filter's stopband Ω_s is $[0, 0.18] \cup [0.42, \pi]$, $[0, 0.38] \cup [0.62, \pi]$, and $[0, 0.58] \cup [0.82, \pi]$ for $\Omega_p = [0.22, 0.38], [0.42, 0.58], [0.62, 0.78]$, respectively. The desired response of the low-pass and band-pass filters is expressed by the following equation.

$$D(\omega_{1,i}, \omega_{2,j}) = \begin{cases} 1, & if \ \sqrt{\omega_{1,i}^2 + \omega_{2,j}^2} \leq \omega_c, \\ 0, & otherwise. \end{cases} \tag{32}$$

$$D(\omega_{1,i}, \omega_{2,j}) = \begin{cases} 1, & if \ \sqrt{\omega_{1,i}^2 + \omega_{2,j}^2} \in \Omega_p, \\ 0, & otherwise. \end{cases} \tag{33}$$

The execution results are shown in Table. I, and Table. II. In Table. I, N_d is the number of non-zero coefficients in the proposed method, N_{CLS} is the number of coefficients of the CLS method, and $E(U)$ and $E(L)$ are the maximum errors in the positive and negative directions, respectively. From Table. I, it can be seen that the proposed method satisfies the maximum error constraint for all the filter specifications, but the maximum error is larger than the maximum error constraint for the CLS method. Experimental results show that the proposed method is superior to the CLS method for filters with equal or greater numbers of multipliers.

V. Conclusion

In this paper, we proposed a two-step approach to design constrained 2D sparse FIR filters. In the first step, the optimal sparse filter coefficients that satisfy the given maximum error constraints are obtained. In the second step, the constrained sparse filter was redesigned to satisfy the results obtained in the first step and to minimize the error with the desired response. The constraints allow the filter coefficients with a value of zero to remain at zero and not exceed the given maximum error. These solutions were obtained by applying ADMM. The proposed filter was able to construct a better filter in terms of maximum error than the CLS method.

In future works, we will improve our sparse filter design algorithm in terms of optimality and computational complexity.

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