

On the Behaviour of Permutation Entropy on Fractional Brownian Motion in a Multivariate Setting

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Abstract—The investigation of qualitative behaviour of the fractional Brownian motion is an important topic for modelling theoretic and real-world applications. Permutation Entropy is a robust and fast approach to quantify the complexity of a time series in a scalar-valued representation. There are numerous studies on the behaviour of Permutation Entropy on fractional Brownian motion. Similarly, Multi-Scale Permutation Entropy is used to study structures on different time scales in a univariate context. Nevertheless, many real-world problems contain multivariate time series. In this paper we investigate the behaviour of Permutation Entropy as well as the behaviour of Multi-Scale Permutation Entropy on fractional Brownian motion – each in the multivariate case. We show that the multivariate results are consistent with known univariate results.

Index Terms—Permutation Entropy, Ordinal Pattern Representations, Fractional Brownian Motion, Multivariate Time Series

I. INTRODUCTION

Time series data are part of many real-world applications, such as weather forecasting, stock markets, energy production, medical recordings, sales and website activities, political or sociological factors. The modelling and prediction of time-dependent dynamical systems with the specific properties of long-range dependence, fractality, or self-similarity is commonly described by fractional Brownian motion, and due to its theory-driven approach successful [1], [2], [3].

For the investigation of the qualitative behaviour of fractional Brownian motion it is valuable to consider efficient mappings from a dynamical system to a set of scalar-valued representations or features capturing specific characteristics. Information theoretic entropies are promising through an encoding that preserves information content [4]. Permutation Entropy is a robust, scalar-valued measure for determining the degree of complexity of time series by analysing the distribution of ordinal patterns. While the associated entropy is low for a deterministic time series, it approaches its maximum value in case of uncorrelated randomness or correlated complexity.

To distinguish between randomness and complexity, an additional measure is necessary. As a complement to Permutation Entropy, Multi-Scale Permutation Entropy captures the complexity of time series on different time scales. On higher time scales, random noise tends to cancel out, resulting in a low entropy measurement, where complex signals retain a high

entropy. In this manner, it is possible to gain a deeper insight into randomness and complexity of a system.

Permutation Entropy on fractional Brownian motion in the univariate case is well understood. Not only the Permutation Entropy, but also the distribution of ordinal patterns of certain lengths result in interesting properties. The distribution of ordinal patterns of lengths two and three yield interesting parameter functions, that can be used for descriptive purposes like autocorrelation. The distribution of simple higher ordinal patterns have irrational values, and depend on the noise distribution. The distribution of ordinal patterns of lengths greater than four does not yield closed formulas and thus, a relationship between autocorrelation and spectrum on the one hand and ordinal patterns on the other hand no longer exist [5].

However, in many fields of applications, multivariate measurements are performed. One example are electrophysiological signals, which are usually not determined from a single electrode, but from multiple electrodes. Further examples for the application of multivariate fractional Brownian motion can be found in economic time series [6], or functional Magnetic Resonance Imaging of several brain regions [7]. This paper contributes to an understanding of the behaviour of multivariate Permutation Entropy, in particular Pooled Permutation Entropy, on multivariate fractional Brownian motion. In addition, we examine multivariate behaviour on different time scales. We show both in a theoretical and in an experimental analysis that the multivariate results are consistent with the univariate case.

II. RELATED WORK

Bandt and Shiha [5] and Zunino [8] significantly contribute to understand the underlying behaviour of Permutation Entropy under known structures such as fractional Brownian motion. They investigate the distribution of ordinal patterns of different orders and, if possible, provide closed formulas for calculation of pattern distributions as well as specific characteristics and relationships (see Section I). Furthermore, Davalos et al. [9] analyse the behaviour of fractional Gaussian noise, the increment process of fractional Brownian motion, under the scope of multi-scaling in an univariate setting.

Nevertheless, the behaviour of Permutation Entropy of fractional Brownian motion was not analysed in the context

of multidimensionality. Amblard et al. [10], [11] summarise theoretical properties of the multivariate fractional Brownian motion and its increments. They provide a characterisation of multivariate fractional Brownian motion by its covariance function and present an algorithm to simulate multivariate fractional Brownian motions used in this paper.

III. PRELIMINARIES

We shortly formalise multivariate fractional Brownian motion and subsequently key concepts of Permutation Entropy, Multi-Scale Permutation Entropy and their extensions to multidimensionality.

A. Multivariate Fractional Brownian Motion

In this paper, we focus on a class of special stochastic processes. A stochastic process or more generally a mathematical object that is similar to itself at all scales is called a fractal. When you zoom in on a fractal, it resembles or looks exactly like the original shape. The mathematical property is called self-similarity. Fractional Brownian motion (fBm) is the unique mean-zero Gaussian process, which is zero at origin and has stationary and self-similar increments. As the name might suggest, the fBm generalises the well-known Brownian motion. The multivariate extension of the fractional Brownian motion requires an extension of self-similarity to multivariate processes first.

Definition 1 (Multivariate self-similar process). A multivariate process $((X^i(t))_{i=1}^m)_{t \in \mathbb{R}} = (X^1(t), \dots, X^m(t))_{t \in \mathbb{R}}$ with variable-dimension $m \in \mathbb{N}$ is called self-similar, if there exists a vector $H = (H_1, \dots, H_m)$ with $H_i \in (0, 1)$ for $i = 1, \dots, m$ such that for any $a > 0$ it is

$$(X_1(at), \dots, X_p(at))_{t \in \mathbb{R}} \sim (a^{H_1} X_1(t), \dots, a^{H_m} X_m(t))_{t \in \mathbb{R}},$$

where \sim denotes the equality of finite-dimensional distributions.

With Definition 1, multivariate fractional Brownian motion is defined as follows:

Definition 2 (Multivariate fractional Brownian motion (mfBm) [11]). An m -multivariate process $((X^i(t))_{i=1}^m)_{t \in \mathbb{R}}$ is called *multivariate fractional Brownian motion* (m -mfBm or mfBm) $\mathbf{B}_H^m(t)$ with Hurst parameter $H \in \mathbb{R}^m$ with $H_i \in (0, 1)$ for $i = 1, \dots, m$ if it is

- 1) Gaussian,
- 2) self-similar with Hurst parameter H and it has
- 3) stationary increments, i.e., $\mathbf{B}_H^m(t) - \mathbf{B}_H^m(s) \sim \mathbf{B}_H^m(t-s)$.

Multivariate self-similarity imposes many constraints on the covariance structure of mfBm [12]. The covariance structure is characterised by parameters $\sigma_i > 0$, $\rho_{ij} \in (-1, 1)$ and $\eta_{ij} \in \mathbb{R}$ for $i, j = 1, \dots, m$. Parameter $\sigma_i > 0$ is the standard deviation of the i -th variable at time 1. Parameter $\rho_{ij} = \rho_{ji}$ is the correlation coefficient between the variables i and j at time 1. Parameters $\eta_{ij} = -\eta_{ji}$ are antisymmetric and linked with the time-reversibility of mfBm.

Multivariate fractional Brownian motion can be characterised by its covariances and cross-covariances of variables m as follows.

Lemma 1 (Covariance Function of mfBm [10]). The mfBm $\mathbf{B}_H^m(t)$ is marginally a fBm, such that the covariance function of the i -th variable $B_{H_i}^i$ of mfBm is as in the univariate case

$$Cov(B_{H_i}^i(s), B_{H_i}^i(t)) = \frac{\sigma_i^2}{2} (|s|^{2H_i} + |t|^{2H_i} - |t-s|^{2H_i}), \quad (1)$$

where $\sigma_i^2 = Var(B_{H_i}^i(1))$. The cross-covariances of mfBm for all $(i, j) \in \{1, \dots, m\}^2$ and $i \neq j$ are given by:

$$Cov(B_{H_i}^i(s), B_{H_j}^j(t)) = \frac{\sigma_i \sigma_j}{2} (w_{ij}(-s) + w_{ij}(t) - w_{ij}(t-s)), \quad (2)$$

where the function $w_{ij}(h)$ is defined by

$$w_{ij}(h) = \begin{cases} (\rho_{ij} - \eta_{ij} \text{sign}(h)) |h|^{H_i+H_j} & \text{for } H_i + H_j \neq 1, \\ \rho_{ij} |h| + \eta_{ij} h \log |h| & \text{for } H_i + H_j = 1. \end{cases} \quad (3)$$

Moreover, the setting of $\rho_{ij} = 1$ and $\eta_{ij} = 0$ in Eq. (2) and Eq. (3) in the cross-covariance function is matching with the covariance function in Eq. (1). For $m = 1$, Definition 2 matches the univariate fBm. In case $H = 1/2$, fBm corresponds to the ordinary Brownian motion. In case $H > 1/2$, the process has a persistence property and positively correlated increments, i.e., an upward jump is more likely followed by another upward jump and vice versa. For $H \rightarrow 1$, the process becomes smoother, less irregular and more trendy. In case $H < 1/2$, the process has negatively correlated increments and an anti-persistence property.

B. Multivariate Permutation Entropy

For the investigation of the qualitative behaviour of mfBm in this paper Permutation Entropy is used. To calculate entropies of time series it is necessary to encode the sequence of real-valued measurements in a series values into a sequence of symbols. As far as current research is concerned, there are two general approaches of symbolisation. On the one hand, classical symbolisation approaches use threshold values and data range partitioning for symbol assignment such as the well-know Symbolic Aggregate ApproXimation (SAX) representation introduced by Chiu et al. [13]. On the other hand, ordinal pattern symbolisation approaches based on an idea of Bandt and Pompe [14] use an encoding of up and down movements in a time series. The formalism and the advantages of the ordinal symbolisation scheme are introduced as follows. Compared to the previously used capitalisation X_i for random variables, we use a small notation x_i for observed random variables, also called events or paths.

Ordinal patterns describe the total order between two or more neighbours, encoded by permutations.

Definition 3 (Univariate Ordinal Pattern). A vector $(x_1, \dots, x_d) \in \mathbb{R}^d$ has *ordinal pattern* $(r_1, \dots, r_d) \in \mathbb{N}^d$ of order $d \in \mathbb{N}$ if $x_{r_1} \geq \dots \geq x_{r_d}$ and $r_{l-1} > r_l$ in the case $x_{r_{l-1}} = x_{r_l}$.

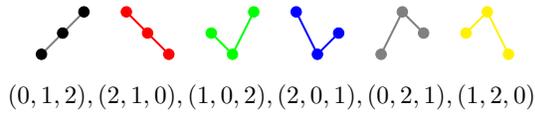


Fig. 1. All possible ordinal patterns of order $d = 3$.

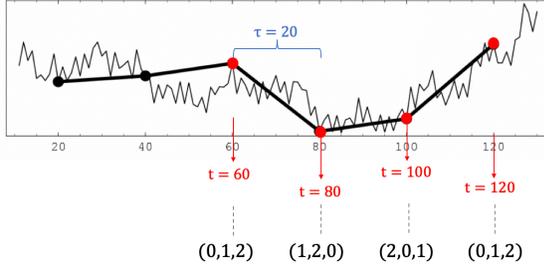


Fig. 2. Ordinal pattern determination of order $d = 3$ and time delay $\tau = 20$ in a univariate time series.

Note that equality of two values within a pattern is not allowed. In this case, for example, the newer value is replaced with a smaller value. Fig. 1 shows all possible ordinal patterns of order $d = 3$ of a vector (x_1, x_2, x_3) . To symbolise a time series $(x_1, x_2, \dots, x_T) \in \mathbb{R}^T$ each time point $t \in \{d, \dots, T\}$ is assigned its ordinal pattern of order d . The order d is chosen much smaller than the length T of the time series to look at small windows in a time series and their distributions of up and down movements. To access the overarching trend, delayed behaviour is of interest. The time delay $\tau \in \mathbb{N}$ is the delay between successive points in the symbol sequences. Different delays show different details of the structure of a time series. Fig. 2 visualises the ordinal pattern determination of order $d = 3$ and time delay $\tau = 20$ of four different time points in a univariate time series.

The ordinal approach has notable advantages in its practical application. First of all, the method is conceptually simple. Second, it is not necessary to have previous knowledge about the data range or type of time series. Third, the ordinal approach supports robust and fast implementations [15], [16]. Fourth, it allows for an easier estimation of a good symbolisation scheme compared to classical approaches [17], [18].

Not the ordinal patterns themselves, but their distributions in different parts of a *univariate time series* $(x_t)_{t=1}^T$ are of interest. Thus, each pattern is identified with exactly one of the *ordinal pattern symbols* $j = 1, 2, \dots, d!$. Using the distribution of the ordinal pattern symbols, its entropy can be calculated using the well-known formula of (Shannon) entropy resulting in the Definition of Permutation Entropy.

Definition 4 (Permutation Entropy [14]). The *Permutation Entropy* (PE) of order $d \in \mathbb{N}$ and delay $\tau \in \mathbb{N}$ of a univariate time series $\mathbf{x} = (x_t)_{t=1}^T$, $T \in \mathbb{N}$ is defined by

$$\text{PE}_{d,\tau}(\mathbf{x}) = - \sum_j^{d!} p_j^{\tau,d} \ln p_j^{\tau,d}, \quad (4)$$

where

$$p_j^{\tau,d} = \frac{\#\{t | (x_{t-(d-1)\tau}, \dots, x_{t-\tau}, x_t) \text{ has pattern } j\}}{T - (d-1)\tau} \quad (5)$$

is the relative frequency of ordinal pattern j in the time series.

Depending on the area of research, entropy is a measure for quantifying inhomogeneity, complexity, uncertainty or unpredictability. For time series with maximum random ordinal pattern symbols (resulting in a uniform pattern distribution due to uniqueness), PE is $\ln(d!)$. For a time series with regular pattern, e.g., in case of monotony, PE is equal to zero [4].

Nevertheless, in real-world applications we often have to deal with high-dimensional multivariate time series. A *multivariate time series* $\mathbf{X} = ((x_t^i)_{i=1}^m)_{t=1}^T$ has more than one time-dependent variable. Each variable x^i for $i \in 1, \dots, m$ depends not only on its past values but also has some dependency on other variables. Considering two time points $(x_t^i)_{i=1}^m$ and $(x_{t+1}^i)_{i=1}^m$ with m variables, it is not possible to establish a total order between them. A total order is only possible if $x_t^i > x_{t+1}^i$ or $x_t^i < x_{t+1}^i$ for all $i \in 1, \dots, m$. Therefore, there is no trivial generalisation of the PE algorithm to the multivariate case. There are numerous studies that deal with the multivariate version of PE [19]. In the following we use the original and most popular approach by Keller and Lauffer [20].

The idea of this approach is to use marginal relative frequencies of $d!$ ordinal patterns regarding all m variables as input for entropy calculation. For each variable $i = 1, \dots, m$ and for each ordinal pattern $j = 1, \dots, d!$, all time steps $s \in [d\tau - \tau + 1, T]$, for which the variable-time pair (i, s) has the ordinal pattern j , are counted. Each variable has a total count of $\delta := T - (d\tau - \tau)$ ordinal patterns. The relative frequencies $p_{ij}^{\tau,d}$ obtained after dividing the counts by $m \cdot \delta$ are stored in a *pooling matrix* $P \in (0, 1)^{m \times d!}$, which reflects the distribution of ordinal patterns in the multivariate time series across its m variables. It is $\sum_{i=1}^m \sum_{j=1}^{d!} p_{ij}^{\tau,d} = 1$. Assume that the marginal relative frequencies $p_j^{\tau,d} = \sum_{i=1}^m p_{ij}^{\tau,d}$ for $j = 1, \dots, d!$ do not vanish. If they vanish set the value close to zero.

Definition 5 (Pooled Permutation Entropy [20]). The *Pooled Permutation Entropy* (PPE) of a multivariate time series $\mathbf{X} = ((x_t^i)_{i=1}^m)_{t=1}^T$ is defined as the Permutation Entropy of the marginal relative frequencies $p_j^{\tau,d} = \sum_{i=1}^m p_{ij}^{\tau,d}$ for $j = 1, \dots, d!$ describing the distribution of the ordinal pattern and can be calculated as

$$\text{PPE}_{d,\tau}(\mathbf{X}) = - \sum_j^{d!} p_j^{\tau,d} \ln p_j^{\tau,d}. \quad (6)$$

In literature, PPE is often referred to as Multivariate Permutation Entropy (MPE). To avoid confusion with other multivariate versions, we use the original naming. For example, PPE is successfully used in analysing electroencephalography (EEG) signals, because cross-channel regularities between spatial-distant variables, i.e., on different hemispheres and/or in different areas, can be extracted by long-range spatial nonlinear correlations [20].

C. Multivariate Multi-Scale Permutation Entropy

Such as biological and physiological time series, as well as time series from other areas, often contain complex structural correlations over several spatio-temporal levels (scales), which need to be uncovered. PE or PPE obtain a maximum entropy on completely random time series as well as series with complex structural correlations. In order to uncover multi-scale structural correlations, the Multi-Scale Entropy (MSE) proposed by Costa et al. [21] provides a systematic procedure with which a small complexity value can be assigned to both completely predictable or completely random uncorrelated time series. In contrast, correlated processes across different scales have a high complexity value. Morabito et al. [22] extend the concept of MSE to ordinal patterns, in both the univariate and multivariate case.

For the consideration of different scales of the time series and an associated definition, a coarse-grained procedure is used: From the original time series, several consecutive time data points are averaged within a non-overlapping time window of scaling length ϵ , also called *scaling factor*. Each element of the coarse-grained time series $\mathbf{y} = (y_l^{(\epsilon)})_{l=1}^{T/\epsilon}$ is calculated as:

$$y_l^{(\epsilon)} = \frac{1}{\epsilon} \sum_{t=(l-1)\epsilon+1}^{l\epsilon} x_t \quad (7)$$

for $1 \leq l \leq \frac{T}{\epsilon}$.

Definition 6 (Multi-Scale Permutation Entropy [22]). The *Multi-Scale Permutation Entropy* (MSPE) of order $d \in \mathbb{N}$ and delay $\tau \in \mathbb{N}$ of a univariate time series $\mathbf{x} = (x_t)_{t=1}^T$, $T \in \mathbb{N}$ is defined as PE of its coarse-grained time series $\mathbf{y} = (y_l^{(\epsilon)})_{l=1}^{T/\epsilon}$, that is

$$\text{MSPE}_{d,\tau,\epsilon}(\mathbf{x}) = \text{PE}_{d,\tau}(\mathbf{y}). \quad (8)$$

The multivariate case is analogous: Per variable i in an multivariate time series, several consecutive time data points are averaged within a non-overlapping time window of the scaling length ϵ . Each element of the coarse-grained time series $\mathbf{Y} = ((y_{i,l}^{(\epsilon)})_{l=1}^{T/\epsilon})_{i=1}^m$ is calculated as:

$$y_{i,l}^{(\epsilon)} = \frac{1}{\epsilon} \sum_{t=(l-1)\epsilon+1}^{l\epsilon} x_{i,t} \quad (9)$$

for all $i = 1, \dots, m$ and $1 \leq l \leq \frac{T}{\epsilon}$.

Definition 7 (Multivariate Multi-Scale Permutation Entropy [22]). The *Multivariate Multi-Scale Permutation Entropy* (MMSPE) of order $d \in \mathbb{N}$ and delay $\tau \in \mathbb{N}$ of a multivariate time series \mathbf{X} is defined as PPE of its coarse-grained time series \mathbf{Y} , that is

$$\text{MMSPE}_{d,\tau,\epsilon}(\mathbf{X}) = \text{PPE}_{d,\tau}(\mathbf{Y}). \quad (10)$$

MSPE and MMSPE are calculated on different time scales by processing the coarse-grained time series as a function of the scale factor ϵ . The simultaneous utilisation of a multi-scale approach and the consideration of multiple variables of

the time series facilitates the assessment of the complexity of the underlying dynamical system.

As far as current research is concerned, the behaviour of MMSPE and PPE on mfBm is unknown and therefore subject of this paper.

IV. POOLED PERMUTATION ENTROPY APPLIED TO MULTIVARIATE FRACTIONAL BROWNIAN MOTION

In the following, we investigate the behaviour of PPE on mfBm in a theoretical as well as experimental setting. For this purpose, we use the univariate results by Bandt and Shiha [5], who show that for univariate fBms the ordinal patterns of order $d = 2$ are equally distributed, more specifically

$$p_{12}^\tau = p_{21}^\tau = 1/2 \quad (11)$$

for all τ . Furthermore, the distribution of ordinal patterns of order $d = 3$ of univariate fBms is given by

$$p_{123}^\tau = \frac{1}{\pi} \arcsin 2^{H-1} \quad (12)$$

for all τ . For a Gaussian process with stationary increments, that includes fBm, it is

$$p_j^\tau = \begin{cases} u & \text{if } j = (123), (321) \\ (1 - 2u)/4 & \text{otherwise} \end{cases} \quad (13)$$

for all τ .

In the following, we show that PPE of orders $d = 2$ and $d = 3$ is invariant to the number of variables m and the multivariate results are consistent with the univariate case.

A. Theoretical Results of Pooled Permutation Entropy on Multivariate Fractional Brownian Motion

The distribution of univariate ordinal pattern of fBms is well understood and can be transferred to multivariate case as follows.

1) *Pooled Permutation Entropy of Order $d = 2$* : In Eq. (11), the distribution of ordinal patterns of order $d = 2$ is independent of Hurst parameter $H \in (0, 1)$ due to self-similarity as well as independent of delay τ . Using the independence of delay τ , the relative frequencies of ordinal patterns in the pooling matrix P of mfBm are again equally distributed:

$$p_{i,12}^\tau = p_{i,21}^\tau = \frac{1}{2m} \quad (14)$$

for all $i = 1, \dots, m$. When calculating the marginal relative frequencies, the number of variables m is reduced again, that is

$$p_{\cdot j}^\tau = \frac{1}{2} \quad (15)$$

for all $j = 1, \dots, d!$. Overall, PPE of order $d = 2$ of mfBm is thus neither dependent on delay τ or Hurst parameter H , nor on the number of variables m . It is

$$\text{PPE}_2(\mathbf{X}) = -\ln \frac{1}{2}. \quad (16)$$

2) *Pooled Permutation Entropy of Order $d = 3$* : In analogy to Paragraph IV-A1 and using Eq. (12) and Eq. (13), the relative frequencies of ordinal patterns of order $d = 3$ in the pooling matrix P of mfBm is given by

$$p_{ij}^\tau = \begin{cases} \frac{1}{m\pi} \arcsin 2^{H_i-1} & \text{if } j = (123), (321) \\ (1 - \frac{2}{\pi} \arcsin 2^{H_i-1}) / (4m) & \text{otherwise} \end{cases} \quad (17)$$

for all $i = 1, \dots, m$ and τ .

For the calculation of the marginal relative frequencies, we distinguish two cases:

- 1) If $H_i = H_j$ for all i and j , the number of variables m is reduced again and the marginal relative frequencies are the same as the frequencies of univariate ordinal pattern in Eq. (12) and Eq. (13).
- 2) If $H_i \neq H_j$ for all $i = j$, the marginal relative frequencies are given by

$$p_{.j}^\tau = \begin{cases} \sum_{i=1}^m \frac{1}{m\pi} \arcsin 2^{H_i-1} & \text{if } j = (123), (321) \\ \sum_{i=1}^m \frac{(1 - \frac{2}{\pi} \arcsin 2^{H_i-1})}{4m} & \text{otherwise} \end{cases} \quad (18)$$

for all τ and $j = 1, \dots, dl$.

For $\sum_{i=1}^m H_i \rightarrow 1$ it is $p_{(123)}^\tau = 1/2$. For $\sum_{i=1}^m H_i \rightarrow 0$, the marginal relative frequencies are distributed equally, that is $p_{.j}^\tau = 1/6$ for all τ .

The consistency regarding the univariate case is not surprising, but is related to the calculation rule of PPE and the independence of the distribution of ordinal patterns from delay parameter τ . The relative frequencies of ordinal patterns of order $d = 4$ in pooling matrix P as well as the marginal relative frequencies of mfBms can be calculated analogously to the cases already treated. For this purpose, the formulas for the distribution of ordinal patterns can be found in [5]. The frequencies of the univariate ordinal patterns of order $d = 5$ cannot be represented in a closed form [5]. Hereafter, we substantiate the theoretical results in an experimental setting.

B. Experimental Results of Pooled Permutation Entropy on Multivariate Fractional Brownian Motion

All experimental calculations are based on a simulation of mfBm using Lemma 1 and a corresponding algorithm implemented by Amblard et al. [11]. The length $T = 7500$ of mfBms is assumed to be large. For a small length T the estimates of the probabilities for the ordinal patterns differ from the true values of a hypothetical time series of infinite length. For the simulation of general mfBms, the correlation parameters $\rho_{i,j}$ in Lemma 1 are all set to 0.3, resulting in

$$\begin{pmatrix} 1 & 0.3 & \dots & 0.3 \\ 0.3 & 1 & & \vdots \\ \vdots & & \ddots & 0.3 \\ 0.3 & \dots & 0.3 & 1 \end{pmatrix} \in \mathbb{R}^{m \times m}. \quad (19)$$

The parameters $\eta_{i,j}$ are set to $0.1/(1 - H_i - H_j)$, resulting in

$$\begin{pmatrix} 0 & \frac{0.1}{1-H_i-H_j} & \dots & \frac{0.1}{1-H_i-H_j} \\ \frac{-0.1}{1-H_i-H_j} & 0 & & \vdots \\ \vdots & & \ddots & \frac{0.1}{1-H_i-H_j} \\ \frac{-0.1}{1-H_i-H_j} & \dots & \frac{-0.1}{1-H_i-H_j} & 0 \end{pmatrix} \in \mathbb{R}^{m \times m} \quad (20)$$

and for $H_i + H_j = 1$ the parameters $\tilde{\eta}_{i,j}$ are set to 0.1. An example of a simulation of mfBm with $m = 5$ variables, equally spaced Hurst parameters H_i in $[0.2, 0.3]$ and the parameters ρ_{ij} and η_{ij} set as above is shown in Fig. 3(a). Fig. 3(b) and Fig. 3(c) illustrate the independence of PPE of order $d = 2$ on mfBm from the number of parameters m for different combinations of the Hurst parameter H_i . For increasing complexity of mfBm by its Hurst parameters H_i the deviation from the theoretical line (purple) increases. Reference [9] shows that the behaviour is explainable by the limitation of length T of the time series. Since the deviation decreases for a higher number of variables m , which is also based on a higher number of total observable ordinal patterns, increasing deviation can be explained by the length limitation T in the setting. Furthermore, Fig. 3(c) shows the independence of the different Hurst parameters $H_1 \neq H_j$ for $j = 2, 3$. Fig. 3(d) and Fig. 3(e) visualise the independence of PPE of order $d = 3$ on mfBm for different combinations of the Hurst parameter H_i from the number of parameters m . The deviation from the theoretical line (purple) can be explained as above. Fig. 3(e) shows the independence of the number of variables m from the dependence of the Hurst parameter H_i .

V. MULTIVARIATE MULTI-SCALE PERMUTATION ENTROPY APPLIED TO MULTIVARIATE FRACTIONAL BROWNIAN MOTION

In the following, we investigate the behaviour of MMSPE on mfBm in a theoretical as well as experimental setting. We show that MMSPE of orders $d = 2$ and $d = 3$ on mfBms is scale-invariant and independent of the number of variables m .

A. Coarse-Graining on Fractional Brownian Motion

To investigate the behaviour of MMSPE on mfBm, coarse-grained fractional Brownian motion has to be derived and analysed first. By using the Eq. (7) or Eq. (9) on fBm, we define coarse-grained fractional Brownian motion (cfBm) as

$$B_{H_i}^{i,(\epsilon)}(l) = \frac{1}{\epsilon} \sum_{j=1}^{\epsilon} B_{H_i}^i((l-1)\epsilon + j) \quad (21)$$

for $l = 1, \dots, T/\epsilon$ and all $i = 1, \dots, m$.

Since the sum of multiple Gaussian variables is again Gaussian distributed, cfBm is completely characterised by the expected value and the covariance function. We follow the arguments of Davalos et al. [9], who characterise the coarse-grained fractional Gaussian noise (cfGn) by its expected value and its covariance function. As the expected value of a sum

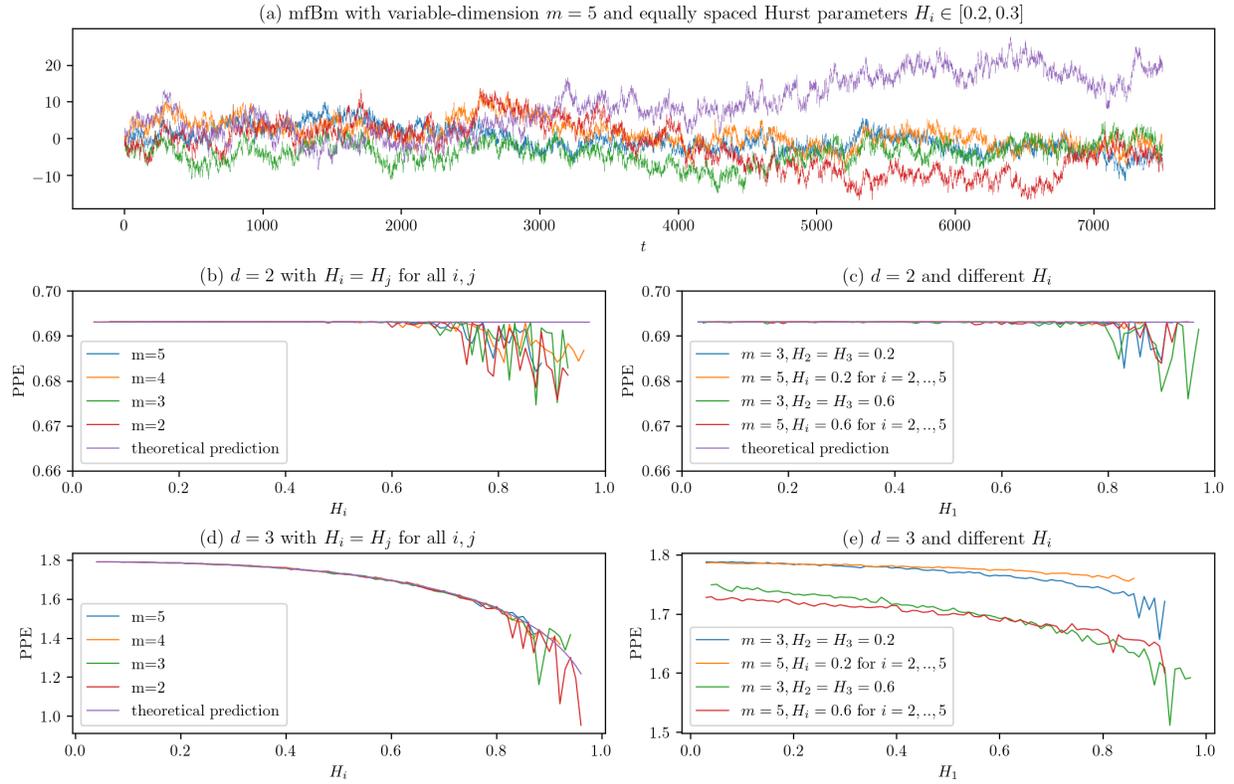


Fig. 3. Simulations of PPE of orders $d = 2$ and $d = 3$, and delay $\tau = 1$ on m -fBm for different Hurst parameters H_i , compared to their theoretical predictions, for signal length $T = 7500$.

of independent random variables is the sum of the expected value of each variable, it is

$$E(B_{H_i}^{i,(\epsilon)}(l)) = E\left(\frac{1}{\epsilon} \sum_{j=1}^{\epsilon} B_{H_i}^i((l-1)\epsilon + j)\right) \quad (22)$$

$$= \sum_{j=1}^{\epsilon} E(B_{H_i}^i((l-1)\epsilon + j)) \quad (23)$$

$$= 0, \quad (24)$$

for all $i = 1, \dots, m$.

Using Eq. (1), self-similarity and stationarity we obtain the variance and covariance of cfBm by

$$\text{Var}(B_{H_i}^{i,(\epsilon)}(l)) = \text{Var}\left(\frac{1}{\epsilon} \sum_{j=1}^{\epsilon} B_{H_i}^i((l-1)\epsilon + j)\right) \quad (25)$$

$$= \sigma_i^2 \epsilon^{2H_i-1} \quad (26)$$

and

$$\text{Cov}(B_{H_i}^{i,(\epsilon)}(l), B_{H_i}^{i,(\epsilon)}(m)) \quad (27)$$

$$= \sigma_i^2 \epsilon^{2H_i-1} (|l|^{2H_i} + |m|^{2H_i} - |m-l|^{2H_i}) \quad (28)$$

for all $i = 1, \dots, m$.

The structure of the covariance function is the same as the original fBm for all variables $i = 1, \dots, m$, but with additional information of the scale factor ϵ . Therefore the covariance function of original fBm or mfBm is invariant to the coarse grain procedure. In the following we use the definition of cfBm and its properties to calculate MMSPE.

B. Theoretical Results of Multivariate Multi-Scale Permutation Entropy on Multivariate Fractional Brownian Motion

Since MSPE and MMSPE are based on PE and PPE, respectively, and the underlying distributions in the coarse-grained procedure of cfBm follow those of the original fBm, the distribution of ordinal patterns of orders $d = 2$ of cfBm are the same as in Eq. (11) and $d = 3$ in Eq. (12). Thus they are scale-invariant. The results of the relative frequencies of ordinal patterns in Eq. (14) and Eq. (17) for the pooling matrices of mfBms as well as the marginal relative frequencies in Eq. (15) and Eq. (18) can be adopted. Thus, the MSPE and MMSPE of the mfBm are scale-invariant and independent of the number of variables m . Any time-scale dependence of the MMSPE is not related to the properties of mfBm, but only results from the fact that the observed data are not infinitely long. A detailed analysis of the length constraints with respect to fractional Gaussian noise is presented in [9]. Hereafter, we substantiate the theoretical results in an experimental setting.

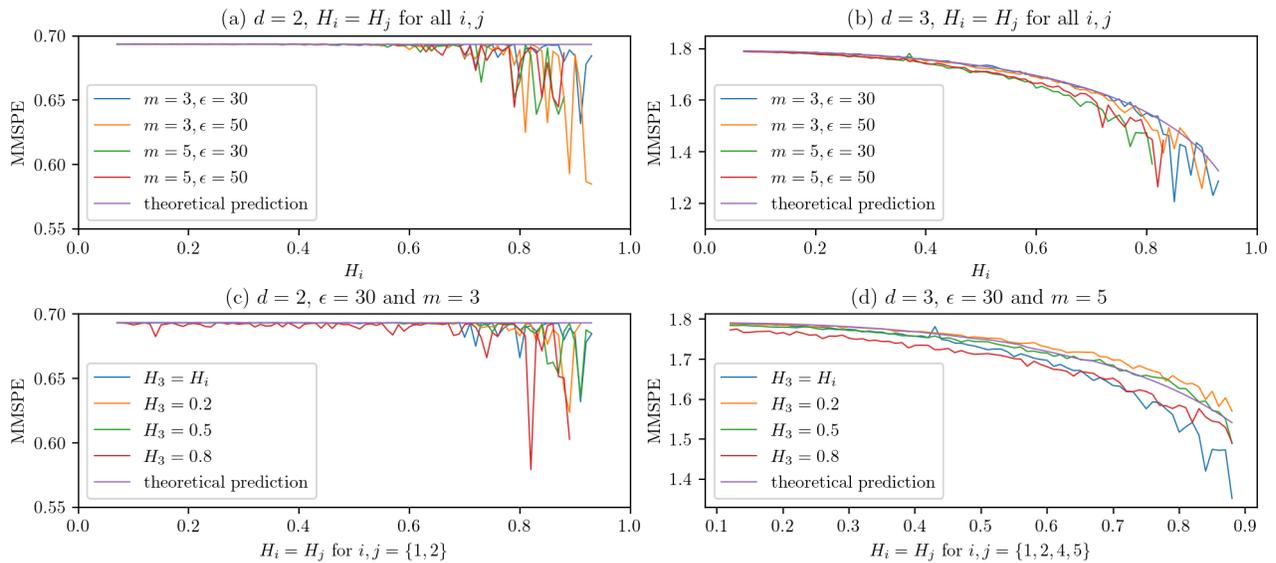


Fig. 4. Simulations of MMSPE of orders $d = 2$ and $d = 3$, and delay $\tau = 1$ on m -fBm for different Hurst parameters H_i , compared to their theoretical predictions, for signal length $T = 10000$.

C. Experimental Results of Multivariate Multi-Scale Permutation Entropy on Multivariate Fractional Brownian Motion

As in Section IV-B, all experimental calculations are based on a simulation of mfBMs using Lemma 1, the corresponding algorithm implemented by Amblard et al. [11], the parameter setting described in Section IV-B, and the length $T = 10000$ of the mfBMs is assumed to be large. Fig. 4 illustrates the behaviour of MMSPE of orders $d = 2$ and $d = 3$ on mfBm with different scale factors ϵ , different variable dimensions m and different Hurst parameters H_i . Again, with increasing Hurst parameters H_i deviations from the theoretical prediction increase due to the length restriction in the experimental setup.

In Fig. 4(a) there is no difference between the behaviour of MMSPE with different scale factors $\epsilon \in \{1, 30, 50\}$ and different dimensions $m \in \{3, 5\}$, but same Hurst parameters $H_i = H_j$ for all $i = j$. All calculations run along the theoretical prediction, so that both the first independence of the MMSPE of order $d = 2$ from the scaling factor and the second independence of the MMSPE of order $d = 2$ from the number of variables m are demonstrated experimentally. The same is true for Fig. 4(b), where the case $d = 3$ is considered.

In Fig. 4(c) and (d) the scale factor is set to $\epsilon = 30$, and the variable number to $m \in \{3, 5\}$. In Fig. 4(c), MMSPE for different Hurst parameters H_i is also analogous to the theoretical prediction, demonstrating the third independence from the Hurst parameters H_i in case of $d = 2$. In Fig. 4(d), MMSPE has a shift up and down for varying Hurst parameters H_i , demonstrating an existing dependency on the Hurst parameters H_i (see Eq. (18) or Fig. 3(e)) in case of $d = 3$. Thus, the experimental results underpin our theoretical findings.

VI. CONCLUSION AND FUTURE WORK

In this paper, we presented a theoretical and experimental analysis of the behaviour of Pooled Permutation Entropy and Multivariate Multi-Scale Permutation Entropy on multivariate fractional Brownian motion. We proved that the behaviour of the PPE and MMSPE of order $d = 2$ on the mfBm is analogous to the univariate case and thus independent of all parameters, especially in the number of variables m . In case of order $d = 3$, the behaviour of PPE and MMSPE on mfBm is only dependent on the Hurst parameter $H \in \mathbb{R}^m$. We showed that PPE and MMSPE do not reveal relevant structures on mfBm with infinite length, although the long and short memory correlations generate a very complex behaviour. In analogy to the univariate case, the intrinsic properties and complex behaviour of mfBMs are not captured by representations like PPE or MMSPE. Further analyses are necessary.

In general, the application of PPE and MMSPE in real-world challenges are promising statistical tools to quantify the complexity even in the multivariate case. However, the application of PPE and MMSPE does not take possible cross-correlations of simultaneous movement patterns of several variables over time as in Eq. (2) and Eq. (3) into account. Since it is not possible to establish a total ordering between vector-valued time steps, a trivial generalisation of the PE algorithm to the multivariate case is not possible. Further proposals, which deal with a multivariate version of PE and the consideration of correlations, exist, e.g., from Mohr et al. [19] and may contribute to an understanding of multivariate Permutation Entropy on multivariate fractional Brownian motion.

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