Windowed Fractional Fourier Transform on Graphs: Fractional Translation Operator and Hausdorff-Young Inequality

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***Abstract*—Designing transform method to identify and exploit structure in signals on weighted graphs is one of the key challenges in the area of signal processing on graphs. So we need to account for the intrinsic geometric structure of the underlying graph data domain. In this paper we generalize the windowed fractional Fourier transform to the graph setting. First we review the windowed fractional Fourier transform and introduce spectral graph theory. Then we define a fractional translation operator with interesting property for signals on graphs. Moreover, we use the operator to define a windowed graph fractional Fourier transform, and explore the reconstruction formula. Finally, the Hausdorff-Young inequality established on this new transform is obtained.**

I. Introduction

Some application areas such as electricity networks, transportation networks, sensor networks in Fig. 1, and social networks in Fig 2, data sets can naturally be modeled as scalar functions defined on the vertices of graphs. Furthermore, weighted graphs can be used to represent the underlying relations between data locations. As an extremely flexible tool, weighted graph is very important to approximate the data domains of a large class of problems.

A new perspective of graph data is to see them as signals. Graph signal processing mainly includes two basic methods. (i) Discrete Signal Processing on Graph ($\mathrm{DSP_G}$) method: This method is derived from Algebraic Signal Processing (ASP) theory. $\mathrm{DSP_G}$ introduces an adjacency matrix that as a shift operator in ASP [1]. (ii) Graph Signal Processing (GSP) method based on graph Laplacian matrix: This method comes from spectral graph theory [2]. GSP uses graph Laplacian matrix as the basic building block, studying the eigenvalues and eigenvectors of graph Laplacian matrix. Then the graph Laplacian matrix and its eigenbases are used to define the spectrum of the graph signal. The following research of this paper is based on the latter method.

The transform methods is one of the fundamental problems of graph signal processing. Several transform methods have been studied, including graph Fourier transform (GFT) [2-6], graph wavelet transform [7-10], fractional Fourier transform on graphs (GFRFT) [11-13], and graph fractional wavelet transform [13]. Because of weighted graphs are irregular structures that lack a shift-invariant notion of translation, a key component in many signal processing techniques for data on regular Euclidean spaces. Thus, many of the existing transforms cannot be directly applied to signals on graphs in a meaningful manner, and an important challenge is to design novel localized transform techniques that analyze the structure of the data domain. In view of the problem, Shuman, Ricaud and Vandergheynst have been proposed windowed graph Fourier transform (WGFT) [14,15]. The windowed graph Fourier transform is obtained by generalizing the classical windowed Fourier transform to the graph setting. Wang and Li [11,12] generalized the fractional Fourier transform (FRFT) to the graph setting. In this paper, we describe a flexible construction for defining windowed fractional Fourier transform for data defined on the vertices of a weighted graph, which is an extended version of windowed graph Fourier transform [14,15].

The paper is organized as follows. In Section 2, We review the windowed fractional Fourier transform and spectral graph theory. In Section 3, we set our notations for weighted graphs ,and define the translation operator in fractional graph domain. In section 4, we define the windowed graph fractional Fourier transform (WGFRFT). To make the transform more complete, we also give the inverse transform. In Section 5, we study Hausdorff-Young inequality for WGFRFT. The last section concludes this paper.

II. Preliminaries

A. Windowed fractional Fourier transform

In this section, we first review the Windowed fractional Fourier transform (WFRFT), which will be needed throughout the paper.

The FRFT with rotational angle α of $f(t)$ on $t \in \mathbb{R}$ is defined as [16,17]

$$\widehat{f_\alpha}(\omega) = (\mathcal{F}_\alpha f)(\omega) = \int_{\mathbb{R}} f(t) K_\alpha(t,\omega)\mathrm{d}t, \quad \omega \in \mathbb{R}, \quad (1)$$

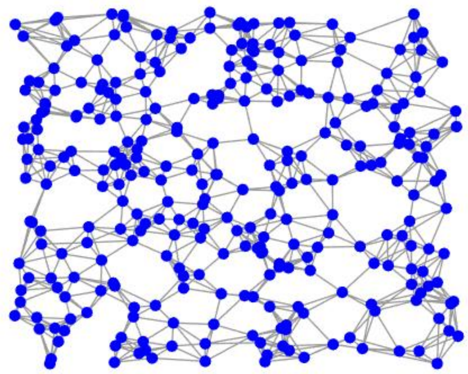

Fig. 1. Sensor network

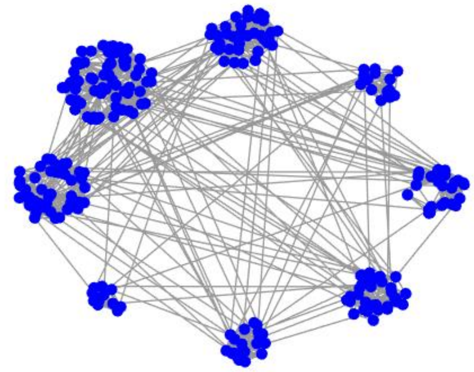

Fig. 2. Social network

where the kernel is shown

$$K_\alpha(t,\omega)=\begin{cases}\sqrt{\frac{1-i\cot\alpha}{2\pi}}e^{\frac{i(t^2+\omega^2)\cot\alpha}{2}} & \\ \times e^{-it\omega\csc\alpha}, & \alpha\neq n\pi, \quad \forall n\in\mathbb{Z},\\ \delta(\omega-t), & \alpha=2n\pi,\\ \delta(\omega+t), & \alpha=(2n+1)\pi.\end{cases} \tag{2}$$

The corresponding inversion formula is given by [16,17]

$$f(t)=\mathcal{F}_\alpha^{-1}[\widehat{f_\alpha}](t)=\int_{\mathbb{R}}\widehat{f_\alpha}(\omega)\overline{K_\alpha(t,\omega)}\mathrm{d}\omega, \quad t\in\mathbb{R}, \tag{3}$$

By multiplying the function $f\in L^2(\mathbb{R})$ with a window function $\psi\in L^2(\mathbb{R})$ before taking the FRFT, the WFRFT is obtained

$$(W_\psi f)(u,\xi)=\int_{\mathbb{R}}f(t)\overline{\psi}(t-u)K_\alpha(t,\xi)\mathrm{d}t, \tag{4}$$

Let $h(t,u)=f(t)\overline{\psi}(t-u)$, then

$$(W_\psi f)(u,\xi)=\int_{\mathbb{R}}h(t,u)K_\alpha(t,\xi)\mathrm{d}t=(\mathcal{F}_\alpha h)(\omega). \tag{5}$$

For every function $f\in L^2(\mathbb{R})$ and $u\in\mathbb{R}$, the translation operator $T_u: L^2(\mathbb{R})\to L^2(\mathbb{R})$ is defined by

$$(T_u f)(t)=f(t-u), \tag{6}$$

Combined with (4) and (5), we know the WFRFT of the function f is equivalent to the FRFT of the function h, where $h(t,u)=(fT_u\overline{\psi})(t)$.

B. Spectral graph theory

Until now, we have showed the definition of WFRFT. It relied on multiplying the signal by a window function to produce a modified signal, expressing the modified signal in the fractional Fourier domain. Our method is to define WFRFT on graphs which depends on generalizing this to graphs, doing so requires the analogue of the fractional Fourier transform for signals defined on the vertices of a weighted graph. As an important tool, spectral graph theory is introduced.

We define undirected, connected, weighted graphs $\mathcal{G}=\{\mathcal{V},\mathcal{E},W\}$, where $\mathcal{V}$ is a finite set of N vertices, $\mathcal{E}$ is a set of edges and W is a weighted adjacency matrix [14,18]. A signal f on a graph $\mathcal{G}$ is a set of real values associated with the nodes of $\mathcal{V}$ and it is a vector of $\mathbb{R}^N$.

$$f:\mathcal{V}\to\mathbb{R}, \tag{7}$$

$$v_n\mapsto f(n). \tag{8}$$

and f can also be written as a real-valued vector

$$f=\begin{bmatrix}f(0) & f(1) & \cdots & f(N-1)\end{bmatrix}^T\in\mathbb{R}^N.$$

For a weighted graph, the degree of each vertex n, denoted as $d(n)$, is the sum of the weights of all the edges incident upon that vertex. We define D as the diagonal degree matrix. The non-normalized graph Laplacian operator $\mathcal{L}=D-W$. $\mathcal{L}$ is a real symmetric matrix and it has a complete set of orthonormal eigenvectors. The orthonormal eigenvectors are denoted by χ_ℓ for $\ell=0,1,\cdots,N-1$. We have

$$\mathcal{L}\chi_\ell=\lambda_\ell\chi_\ell,$$

where λ_ℓ are the associated real, non-negative Laplacian eigenvalues and the eigenvalues are ordered as $0=\lambda_0<\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_{N-1}=\lambda_{max}$.

For any function $f\in\mathbb{R}^N$ defined on the vertices of $\mathcal{G}$, its GFT is defined by [7,14,18]

$$\widehat{f}(\ell)=\langle f,\chi_\ell\rangle=\sum_{n=1}^{N}f(n)\chi_\ell^*(n). \tag{9}$$

The inverse graph Fourier transform (IGFT) is given by

$$f(n)=\sum_{\ell=0}^{N-1}\widehat{f}(\ell)\chi_\ell(n). \tag{10}$$

The Parseval relation of the graph Fourier transform is obtained, that is, for any signals f and g defined on the graph $\mathcal{G}$ we have:

$$\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle, \tag{11}$$

If $f = g$, then

$$\sum_{n=1}^{N} |f(n)|^2 = \|f\|_2^2 = \langle f, f\rangle = \langle \widehat{f}, \widehat{f}\rangle = \|\widehat{f}\|_2^2 = \sum_{\ell=1}^{N-1} |\widehat{f}(\ell)|^2. \tag{12}$$

Similar to the GFT, the graph fractional Laplacian operator $\mathcal{L}_\alpha$ is defined by $\mathcal{L}_\alpha \kappa_\ell = k_\ell \kappa_\ell$. where $0 < \alpha \leq 1$, $k_\ell = \lambda_\ell^\alpha$ and $\kappa_\ell = \chi_\ell^\alpha$ [13].

The GFRFT of any signal $f \in \mathbb{R}^N$ defined on the vertices $\mathcal{V}$ of the graph $\mathcal{G}$ is defined by [13]:

$$\widehat{f}_\alpha(\ell) = \langle f, \kappa_\ell\rangle = \sum_{n=1}^{N} f(n)\kappa_\ell^*(n), \tag{13}$$

The inverse GFRFT is given by

$$f(n) = \sum_{\ell=0}^{N-1} \widehat{f}_\alpha(\ell)\kappa_\ell(n). \tag{14}$$

The Parseval relation of the GFRFT is obtained, that is, for any signals f and g defined on the graph $\mathcal{G}$ we have:

$$\langle f, g\rangle = \langle \widehat{f}_\alpha, \widehat{g}_\alpha\rangle. \tag{15}$$

If $f = g$, then

$$\sum_{n=1}^{N} |f(n)|^2 = \|f\|_2^2 = \langle \widehat{f}_\alpha, \widehat{f}_\alpha\rangle = \|\widehat{f}_\alpha\|_2^2 = \sum_{\ell=1}^{N-1} |\widehat{f}_\alpha(\ell)|^2. \tag{16}$$

III. Fractional translation operator on graph

For any signal $f \in \mathbb{R}^N$ defined on the graph $\mathcal{G}$ and any $i \in \{1, 2, \cdots, N\}$, we can define a fractional translation operator $T_i^\alpha : \mathbb{R}^N \to \mathbb{R}^N$

$$(T_i^\alpha f)(n) = (\sqrt{N})^\alpha \sum_{\ell=0}^{N-1} \widehat{f}_\alpha(\ell)\kappa_\ell^*(i)\kappa_\ell(n). \tag{17}$$

Do GFRFT in (13) to this new operator, we find that it can simplify to a product about the GFRFT of signal f and the eigenvector of graph fractional Laplacian operator.

$$\begin{aligned}\widehat{(T_i^\alpha f)}_\alpha(\ell) &= ((T_i^\alpha f)(n), \kappa_\ell) \\ &= \sum_{n=0}^{N-1} (\sqrt{N})^\alpha \sum_{\ell=0}^{N-1} \widehat{f}_\alpha(\ell)\kappa_\ell^*(i)\kappa_\ell(n)\kappa_\ell^*(n) \\ &= (\sqrt{N})^\alpha \sum_{\ell=0}^{N-1} \widehat{f}_\alpha(\ell)\kappa_\ell^*(i)\end{aligned} \tag{18}$$

According to the above definition, we find an useful property about the bounds of fractional translation operator on graph.

For the convenience of follow-up proof, we give the largest absolute value of the elements of a given graph fractional Laplacian eigenvector by

$$a_\ell = \|\kappa_\ell\|_\infty = \max_{i \in \{1,2,\cdots,N\}} |\kappa_\ell(i)|, \tag{19}$$

and

$$b_i = \max_{\ell \in \{0,1,\cdots,N-1\}} |\kappa_\ell(i)|, \tag{20}$$

Note that

$$M = \max_{\ell \in \{0,1,\cdots,N-1\}} \{a_\ell\} = \max_{i \in \{1,2,\cdots,N\}} \{b_i\}, \tag{21}$$

Lemma 1: For any $f \in \mathbb{R}^N$,

$$|\widehat{f}_\alpha(0)| \leq \|T_i^\alpha f\|_2 \leq (\sqrt{N})^\alpha b_i \|\widehat{f}_\alpha\|_2 \leq (\sqrt{N})^\alpha M \|\widehat{f}_\alpha\|_2. \tag{22}$$

Proof: In line with (16) and (17), we obtain

$$\begin{aligned}\|T_i^\alpha f\|_2^2 &= \sum_{n=1}^{N} \left((\sqrt{N})^\alpha \sum_{\ell=0}^{N-1} \widehat{f}_\alpha(\ell)\kappa_\ell^*(i)\kappa_\ell(n) \right)^2 \\ &= N^\alpha \sum_{\ell=0}^{N-1} \sum_{\ell'=0}^{N-1} \widehat{f}_\alpha(\ell)\widehat{f}_\alpha(\ell')\kappa_\ell^*(i)\kappa_{\ell'}^*(i) \\ &\times \sum_{n=1}^{N} \kappa_\ell(n)\kappa_{\ell'}(n) \\ &= N^\alpha \sum_{\ell=0}^{N-1} |\widehat{f}_\alpha(\ell)|^2 |\kappa_\ell^*(i)|^2 \\ &\leq N^\alpha b_i^2 \|\widehat{f}_\alpha\|_2^2.\end{aligned} \tag{23}$$

By (21) and note that $\kappa_0(i) = (\frac{1}{\sqrt{N}})^\alpha$, we get the result. ■

IV. Windowed graph fractional Fourier transform

Based on the fractional translation operator on graph, we can define a windowed graph fractional Fourier transform (WGFRFT) analogously to (4) and (5) as follows

$$\begin{aligned}W_{\psi,\alpha} f(i,l) &= \sum_{n=1}^{N} (f T_i^\alpha \overline{\psi})(n)\kappa_\ell^*(n) \\ &= \sum_{n=1}^{N} f(n)(\sqrt{N})^\alpha \sum_{\ell'=0}^{N-1} \widehat{\overline{\psi}}_\alpha(\ell')\kappa_{\ell'}^*(i)\kappa_{\ell'}(n)\kappa_\ell^*(n).\end{aligned} \tag{24}$$

Let $h_i(n) = (f T_i^\alpha \overline{\psi})(n)$, then

$$W_{\psi,\alpha} f(i,l) = \sum_{n=1}^{N} h_i(n)\kappa_\ell^*(n) = \widehat{h_{i,\alpha}}(\ell). \tag{25}$$

Let $\Psi_{i,\ell}^*(n) = (\sqrt{N})^\alpha \kappa_\ell^*(n) \sum_{\ell'=0}^{N-1} \widehat{\overline{\psi}}_\alpha(\ell')\kappa_{\ell'}^*(i)\kappa_{\ell'}(n)$, then

$$W_{\psi,\alpha} f(i,l) = \langle f, \Psi_{i,\ell}\rangle. \tag{26}$$

When $\alpha = 1$, the WGFRFT becomes the WGFT [15], that is, the WGFRFT is a generalization of WGFT.

We will show that the inverse formula of the WGFRFT, it can reconstruct a signal corresponding to a given set of transform coefficients, which is useful for signal processing and signal analysis.

Theorem 1: If $\widehat{\widehat{\psi}}_\alpha(0) \neq 0$, then for any signal $f \in \mathbb{R}^N$, we have

$$f(n) = \frac{1}{N^\alpha \|T_i^\alpha \overline{\psi}\|_2^2} \sum_{i=1}^{N} \sum_{\ell=0}^{N-1} W_{\psi,\alpha} f(i,l) \Psi_{i,\ell}(n) \tag{27}$$

Proof: By (14) and (25), we have

$$f(n)(\sqrt{N})^\alpha \sum_{\ell'=0}^{N-1} \widehat{\widehat{\psi}}_\alpha(\ell') \kappa_{\ell'}^*(i) \kappa_{\ell'}(n) = \sum_{\ell=0}^{N-1} W_{\psi,\alpha} f(i,l) \kappa_\ell(n). \tag{28}$$

By multiplying $(\sqrt{N})^\alpha \sum_{i=1}^{N} \sum_{\ell''=0}^{N-1} \widehat{\widehat{\psi}}_\alpha(\ell'') \kappa_{\ell''}(i) \kappa_{\ell''}^*(n)$ on both sides of (28), we obtain

$$\begin{aligned} &f(n)(\sqrt{N})^\alpha \sum_{\ell'=0}^{N-1} \widehat{\widehat{\psi}}_\alpha(\ell') \kappa_{\ell'}(n) (\sqrt{N})^\alpha \sum_{\ell''=0}^{N-1} \widehat{\widehat{\psi}}_\alpha(\ell'') \kappa_{\ell''}^*(n) \\ &\times \sum_{i=1}^{N} \kappa_{\ell'}^*(i) \kappa_{\ell''}(i) \\ &= \sum_{i=1}^{N} \sum_{\ell=0}^{N-1} W_{\psi,\alpha} f(i,l) (\sqrt{N})^\alpha \kappa_\ell(n) \sum_{\ell''=0}^{N-1} \widehat{\widehat{\psi}}_\alpha(\ell'') \kappa_{\ell''}(i) \kappa_{\ell''}^*(n). \end{aligned} \tag{29}$$

Hence

$$N^\alpha f(n) \sum_{\ell'=0}^{N-1} |\widehat{\widehat{\psi}}_\alpha(\ell')|^2 |\kappa_{\ell'}(n)|^2 = \sum_{i=1}^{N} \sum_{\ell=0}^{N-1} W_{\psi,\alpha} f(i,l) \Psi_{i,\ell}(n). \tag{30}$$

According to (23), we have

$$N^\alpha \|T_i^\alpha \overline{\psi}\|_2^2 f(n) = \sum_{i=1}^{N} \sum_{\ell=0}^{N-1} W_{\psi,\alpha} f(i,l) \Psi_{i,\ell}(n) \tag{31}$$

■

V. Hausdorff-Young inequalities

Lemma 2: (Riesz-Thorin interpolation theorem) [19]. Let Γ is a bounded linear operator from ℓ^{p_1} to ℓ^{p_2},and from ℓ^{q_1} to ℓ^{q_2}, there exist constants D_p and D_q such that

$$\|\Gamma f\|_{p_2} \leq D_p \|f\|_{p_1}, \tag{32}$$

and

$$\|\Gamma f\|_{q_2} \leq D_q \|f\|_{q_1}. \tag{33}$$

Then for any $t \in (0,1)$, Γ is also a bounded operator from ℓ^{r_1} to ℓ^{r_2}:

$$\|\Gamma f\|_{r_2} \leq D_r \|f\|_{r_1}, \tag{34}$$

with

$$\frac{1}{r_1} = \frac{t}{p_1} + \frac{1-t}{q_1}, \quad \frac{1}{r_2} = \frac{t}{p_2} + \frac{1-t}{q_2}, \tag{35}$$

and

$$D_r = D_p^t D_q^{1-t}. \tag{36}$$

Theorem 2: (Hausdorff-Young inequalities for fractional translation operator) Let $p, q > 0$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. For any signal $f \in \mathbb{R}^N$ defined on a graph $\mathcal{G}$ and $2 \leq p \leq \infty, 1 \leq q \leq 2$, we have

$$\|T_i^\alpha f\|_p \leq (\sqrt{N})^\alpha M \|f\|_q. \tag{37}$$

Proof: From Lemma 1, we have $\|T_i^\alpha f\|_2 \leq (\sqrt{N})^\alpha M \|f\|_2$. Then, using the proof of Theorem 2 in [19], we obtain

$$\|T_i^\alpha f\|_\infty \leq (\sqrt{N})^\alpha M \|f\|_1. \tag{38}$$

Applying the Riesz-Thorin interpolation theorem with $p_1 = 2$, $p_2 = 2$, $q_1 = 1$, $q_2 = \infty$, $D_p = (\sqrt{N})^\alpha M$, $D_q = (\sqrt{N})^\alpha M$, $t = \frac{2}{q}$, $r_1 = q$ and $r_2 = p$ yields the inequality. ■

In terms of Theorem 2 in [19], we introduce the following lemma:

Lemma 3: Let $p, q > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. For any signal $f \in \mathbb{R}^N$ defined on a graph $\mathcal{G}$ and $1 \leq p \leq 2$, $2 \leq q \leq \infty$, there is

$$\|\widehat{f}_\alpha\|_q \leq M^{1-\frac{2}{q}} \|f\|_p. \tag{39}$$

Theorem 3: (Hausdorff-Young inequalities for WGFRFT) Let $p, q > 0$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, $\psi \in \mathbb{R}^N$ is a window function, for any signal $f \in \mathbb{R}^N$ defined on a graph $\mathcal{G}$ and $1 \leq p \leq 2$, $2 \leq q \leq \infty$, we have

$$\|W_{\psi,\alpha} f(i,l)\|_q \leq (\sqrt{N})^\alpha M^{\frac{2}{p}} \|f\|_p \|\overline{\psi}\|_1. \tag{40}$$

Proof: Follow from (25) and Lemma 3,

$$\begin{aligned} \|W_{\psi,\alpha} f(i,l)\|_q &= \|\widehat{h_{i,\alpha}}\|_q \\ &\leq M^{1-\frac{2}{q}} \|f T_i^\alpha \overline{\psi}\|_p \\ &= M^{1-\frac{2}{q}} \left(\sum_{n=1}^{N} |(f T_i^\alpha \overline{\psi})(n)|^p \right)^{\frac{1}{p}} \\ &= M^{1-\frac{2}{q}} \left(\sum_{n=1}^{N} |f(n|^p |(T_i^\alpha \overline{\psi})(n)|^p \right)^{\frac{1}{p}} \\ &\leq M^{1-\frac{2}{q}} \|T_i^\alpha \overline{\psi}\|_\infty \left(\sum_{n=1}^{N} |f(n|^p \right)^{\frac{1}{p}} \end{aligned} \tag{41}$$

by (38), we obtain

$$\|W_{\psi,\alpha} f(i,l)\|_q \leq (\sqrt{N})^\alpha M^{\frac{2}{p}} \|f\|_p \|\overline{\psi}\|_1. \tag{42}$$

■

VI. Conclusions

In this paper, to multiply a window function to a graph signal, we define a fractional translation operator on graph and display related bounds. Matching with the windowed fractional Fourier transform, we apply this translation operator to design a new transform named windowed graph fractional Fourier transform and its inverse transform. Finally, as an interesting property, we present the Hausdorff-Young inequality for WGFRFT.

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