On Optimal Realizations for All-Pass Fractional Delay Digital Filters

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***Abstract*—This paper discusses synthesis of optimal realizations with respect to finite wordlength effects for all-pass fractional delay digital filters. In particular this paper focuses on mathematical synthesis of optimal realizations based on state-space representation, and reveals that some well-known realizations such as the balanced realization and the normalized lattice structure provide the optimal realization that simultaneously minimizes the L_1/L_2-mixed sensitivity and the L_2-sensitivity. In addition, deviation of frequency responses for the optimal realizations is investigated through some numerical examples.**

I. Introduction

In digital signal processing, fractional delay digital filters are used to approximately realize non-integer sample delays [1]. Practical applications of fractional delay digital filters include sampling rate conversion, non-uniform sampling, signal interpolation, modeling of musical instruments, and software radio. For both FIR and IIR digital filters, a number of design methods on fractional delay filters have been proposed in the literature. Among such methods, this paper focuses on the IIR all-pass fractional delay digital filters [1]. This kind of fractional delay filters is based on the approximation of ideal fractional group delays by means of Thiran's method [2], leading to realization of maximally flat group delay characteristics at zero frequency. In addition, this design method yields the ideal (unity) magnitude response for any given value of the desired fractional delays because of the all-pass property.

The purpose of this paper is to investigate the problem on synthesis of optimal filter structures for the transfer functions of the all-pass fractional delay filters. Up to the present, design of transfer functions of fractional delay digital filters has been well studied in the literature. However, little has been discussed about how the choice of filter structures affects the characteristics of fractional delay digital filters under the finite wordlength environment. As is well known, such finite wordlength effects highly depend on filter structures, and high-accuracy filters can be attained through optimal synthesis of structures for a given transfer function. Therefore, synthesis of optimal structures for the all-pass fractional delay filters is one of the important topics in recent advanced techniques for digital filters.

In this paper, the problem on synthesis of filter structures, mainly from the viewpoint of the coefficient quantization, is addressed by means of the state-space representation. In addition, it is shown that some well-known realizations such as the balanced realization and the normalized lattice structure provide the optimal structure for the all-pass fractional delay filters. Numerical examples demonstrate that the normalized lattice structure attains the highest accuracy for realization of the prescribed fractional delays under coefficient quantization.

The organization of this paper is as follows. Section II gives preliminaries on the all-pass fractional delay digital filters and the state-space representation. Section III investigates the theory on optimal structures of all-pass fractional delay digital filters. Section IV gives numerical examples to evaluate the accuracy of all-pass fractional digital filters over specific filter structures with finite coefficient wordlength.

II. Preliminaries

A. All-pass fractional delay digital filters

The transfer function of the all-pass fractional delay digital filters is described as

$$\begin{aligned} H_{\mathrm{FD}}(z) &= \frac{a_N + a_{N-1}z^{-1} + \cdots + z^{-N}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} \\ a_k &= (-1)^k \begin{pmatrix} N \\ k \end{pmatrix} \prod_{n=0}^{N} \frac{D-N+n}{D-N+k+n}, \\ & \quad 1 \le k \le N \end{aligned} \tag{1}$$

where D is the value of the desired fractional delay that is prescribed by user, and N denotes the order of the transfer function. It is shown in [3] that the stability of $H_{\mathrm{FD}}(z)$ is ensured for any D satisfying $D > N-1$. Noting that $H_{\mathrm{FD}}(z)$ is an all-pass transfer function, it readily follows that the magnitude response becomes unity, i.e.

$$|H_{\mathrm{FD}}(e^{j\omega})| = 1 \tag{2}$$

holds for any D, where ω denotes the normalized angular frequency. Figure 1 shows an example of the group delay characteristics of $H_{\mathrm{FD}}(z)$ for $D = 3.1$, $D = 3.4$ and $D = 3.8$, respectively. As shown in this figure, the maximally flat group delays at zero frequency are obtained for all D, leading to high-accuracy approximations of the desired fractional delays in the low frequency region.

B. State-space representation

It is well known that the transfer function $H(z)$ of an N-th order digital filter can be represented in the following state-space form

$$H(z) = d + \boldsymbol{c}(z\boldsymbol{I}_N - \boldsymbol{A})^{-1}\boldsymbol{b} \tag{3}$$

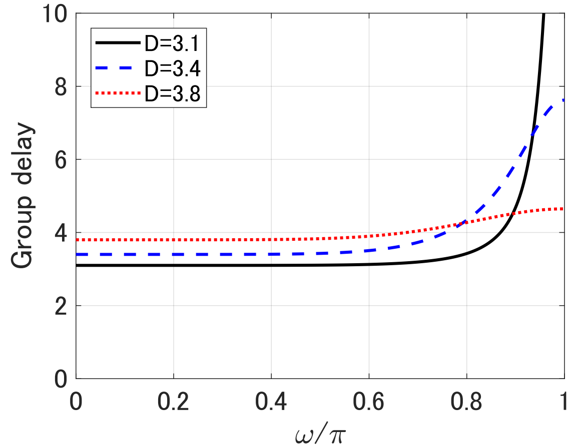

Fig. 1. Example of group delay characteristics of all-pass fractional delay filters.

where $\boldsymbol{I}_N$ denotes the $N \times N$ identity matrix and $\boldsymbol{A} \in \Re^{N\times N}$, $\boldsymbol{b} \in \Re^{N\times 1}$, $\boldsymbol{c} \in \Re^{1\times N}$ and $d \in \Re^{1\times 1}$ are real-valued coefficients of the state-space representation. The time-domain input-output relationship is given in terms of this set of coefficients as

$$\boldsymbol{x}(n+1) = \boldsymbol{A}\boldsymbol{x}(n) + \boldsymbol{b}u(n) \tag{4}$$
$$y(n) = \boldsymbol{c}\boldsymbol{x}(n) + du(n) \tag{5}$$

where $u(n)$ and $y(n)$ are the input signal and the output signal, and $\boldsymbol{x}(n) \in \Re^{N\times 1}$ is the state vector that corresponds to outputs of delay elements in the filter.

It is important to note that the set $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ is non-unique for a given transfer function $H(z)$, and that the filter accuracy with respect to the finite wordlength effects depends on the choice of $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$. In other words, an appropriate choice of $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$, which means appropriate synthesis of filter structure, attains high accuracy under the finite wordlength environment.

An example of such high-accuracy structures is the balanced realization [4]. This realization makes use of the two positive definite matrices $\boldsymbol{K}$ and $\boldsymbol{W}$ that are respectively defined as

$$\boldsymbol{K} = \sum_{n=0}^{\infty} \boldsymbol{A}^n \boldsymbol{b}\boldsymbol{b}^T (\boldsymbol{A}^T)^n \tag{6}$$
$$\boldsymbol{W} = \sum_{n=0}^{\infty} (\boldsymbol{A}^T)^n \boldsymbol{c}^T \boldsymbol{c} \boldsymbol{A}^n. \tag{7}$$

These matrices $\boldsymbol{K}$ and $\boldsymbol{W}$ are respectively called the controllability Gramian and the observability Gramian. The balanced realization is given in such a manner that the set $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ satisfies

$$\boldsymbol{K} = \boldsymbol{W} = \boldsymbol{\Theta} \tag{8}$$

where $\boldsymbol{\Theta}$ denotes the positive diagonal matrix of which diagonal elements are given by the positive square roots of the eigenvalues of the matrix product $\boldsymbol{K}\boldsymbol{W}$. This realization is known to be the optimal realization in that the following cost function is minimized [5]:

$$S_{L_1/L_2} = \mathrm{tr}(\boldsymbol{K})\mathrm{tr}(\boldsymbol{W}) + \mathrm{tr}(\boldsymbol{K}) + \mathrm{tr}(\boldsymbol{W}). \tag{9}$$

This function is called the L_1/L_2-mixed sensitivity, which is used as a measurement of the coefficient sensitivity of digital filters. Hence the balnced realization is an optimal structure with respect to the coefficient quantization.

Another example of high-accuracy structures is the minimum L_2-sensitivity realization, which is a state-space representation that minimizes the following cost function

$$S_{L_2} = \mathrm{tr}(\boldsymbol{K})\mathrm{tr}(\boldsymbol{W}) + \mathrm{tr}(\boldsymbol{K}) + \mathrm{tr}(\boldsymbol{W}) + 2\sum_{i=1}^{\infty} \mathrm{tr}(\boldsymbol{K}_i)\mathrm{tr}(\boldsymbol{W}_i) \tag{10}$$

where $\boldsymbol{K}_i$ and $\boldsymbol{W}_i$ for $i \geq 1$ are called the general controllability/observability Gramians [6], and they are simply calculated from $\boldsymbol{A}$, $\boldsymbol{K}$ and $\boldsymbol{W}$ as [7]

$$\boldsymbol{K}_i = \frac{1}{2}\left(\boldsymbol{A}^i\boldsymbol{K} + \boldsymbol{K}(\boldsymbol{A}^T)^i\right) \tag{11}$$
$$\boldsymbol{W}_i = \frac{1}{2}\left((\boldsymbol{A}^T)^i\boldsymbol{W} + \boldsymbol{W}\boldsymbol{A}^i\right). \tag{12}$$

In general, minimization of (10) is very difficult because of the existence of general controllability/observability Gramians. In the case of all-pass filters, however, it is shown in [8] that this minimization problem is simplified and the minimum L_2-sensitivity realization is given by the balanced realization.

III. Optimal Structures of All-Pass Fractional Delay Digital Filters

Based on the theory introduced in the previous section, the optimal realization of all-pass fractional delay digital filters can be easily obtained in terms of the state-space representation. The following theorem summarizes this fact.

Theorem 1: For any transfer function of all-pass fractional delay digital filter, its optimal realization is given by the balanced realization, which simultaneously minimizes the L_1/L_2-mixed sensitivity and the L_2-sensitivity.

This theorem is easily proved from the theory of the previous section, and thus omitted here.

In addition to Theorem 1, this paper presents the following theorem that shows the link between the balanced form and the normalized lattice structure.

Theorem 2: For any transfer function of all-pass fractional delay digital filter, its optimal realization is given by the normalized lattice structure (i.e., four-multiplier lattice structure) [9], which simultaneously minimizes the L_1/L_2-mixed sensitivity and the L_2-sensitivity.

Theorem 2 is also easily proved by using the theory of state-space digital filters. First, it is proved in [10] that the controllability Gramian of the normalized lattice structure becomes the identity matrix, i.e.

$$\boldsymbol{K} = \boldsymbol{I}_N \tag{13}$$

holds for the normalized lattice structure. Second, it is also proved in [10] that the controllability/observability Gramians of any all-pass filter satisfy

$$\boldsymbol{K} = \boldsymbol{W}^{-1}. \tag{14}$$

Hence Eqs. (13) and (14) reveal that any all-pass filter with the normalized lattice structure satisfies

$$\boldsymbol{K} = \boldsymbol{W} = \boldsymbol{I}_N, \tag{15}$$

showing that the normalized lattice structure is equivalent to the balanced realization in the case of all-pass filters. This result and Theorem 1 lead to the conclusion that the normalized lattice structure becomes the optimal realization for any all-pass fractional delay digital filter and complete the proof.

Now, Theorems 1 and 2 tell us that, in the case of all-pass fractional delay digital filters, the normalized lattice structure theoretically gives the highest accuracy with respect to the coefficient quantization error. This will be demonstrated by numerical examples of the next section.

Remark 1: Although any all-pass fractional delay filter with the normalized lattice structure theoretically provides the highest accuracy with respect to the coefficient quantization error, this structure has a drawback that the all-pass property (i.e. the unity magnitude response for all input frequencies) is violated under coefficient quantization. On the other hand, in other lattice structures such as one-multiplier and two-multiplier lattice structures as well as the standard direct-form structure, the all-pass property is preserved even if the filter coefficients are quantized [11]. Hence the numerical examples of the next section will also demonstrate this fact.

IV. Numerical Examples

The numerical examples to be presented here evaluate the accuracy of all-pass fractional delay digital filters of which coefficients are quantized. The filter structures to be considered here are the direct-form structure, the 2-multiplier lattice structure, and the normalized lattice structure. For evaluation, three cases of ideal group delays are considered, where the value of D is set to be 8.2, 8.5 and 8.8, respectively. Note that ninth-order transfer functions are used in all of these cases. The corresponding filter coefficients of infinite wordlength, which are obtained by substituting the values of D and $N = 9$ into Eq. (1), are summarized in Table I.

TABLE I
Coefficients of all-pass fractional delay filters.

	$D = 8.2$	$D = 8.5$	$D = 8.8$
a_1	0.7826	0.4737	0.1837
a_2	-6.1381×10^{-2}	-9.0226×10^{-2}	-5.4422×10^{-2}
a_3	1.5345×10^{-2}	2.7460×10^{-2}	1.9370×10^{-2}
a_4	-4.1508×10^{-3}	-8.2380×10^{-3}	-6.3559×10^{-3}
a_5	1.0062×10^{-3}	2.1358×10^{-3}	1.7502×10^{-3}
a_6	-1.9841×10^{-4}	-4.4188×10^{-4}	-3.7842×10^{-4}
a_7	2.9091×10^{-5}	6.7199×10^{-5}	5.9534×10^{-5}
a_8	-2.7834×10^{-6}	-6.6181×10^{-6}	-6.0243×10^{-6}
a_9	1.2946×10^{-7}	3.1515×10^{-7}	2.9332×10^{-7}

A. Evaluation of group delay characteristics

In this subsection the accuracy of quantized all-pass fractional delay filters are evaluated in terms of the group delay characteristics. For the three all-pass fractional delay filters specified in Table I, coefficient quantization is performed to each structure. The worlength of all coefficients is set to be 7 fractional bits. The resultant group delay characteristics are shown in Figs. 2, 3 and 4. The group delay errors for each structure are also evaluated for quantitative evaluation of the accuracy with respect to the coefficient quantization. This evaluation result is summarized in Table II. Here, the error is measured as the Mean Square Error (MSE) between the group delay characteristics of ideal (infinite wordlength) and quantized filters. Calculation of the MSE is based on

TABLE II
Group delay MSEs of quantized filters.

	$D = 8.2$	$D = 8.5$	$D = 8.8$
Direct-form	3.3088×10^{-3}	1.2716×10^{-3}	4.6219×10^{-4}
2-multiplier lattice	8.3896×10^{-4}	6.7031×10^{-4}	4.7079×10^{-4}
Normalized lattice	8.2751×10^{-4}	4.7079×10^{-4}	4.5697×10^{-4}

$$\text{MSE} = \frac{1}{M}\sum_{k=1}^{M}\left(\tau_{\text{FD}}(\omega_k) - \tau'_{\text{FD}}(\omega_k)\right)^2 \tag{16}$$

where the normalized angular frequency ω for $0 \leq \omega \leq \pi$ is equally discretized into M points $\omega_1, \omega_2, \cdots, \omega_M$, and $\tau_{\text{FD}}(\omega_k)$ and $\tau'_{\text{FD}}(\omega_k)$ respectively correspond to the ideal and the quantized group delays evaluated at ω_k. Table II shows that the normalized lattice structure attains the smallest error (i.e. the highest accuracy) under the coefficient quantization.

B. Evaluation of magnitude responses

As stated in Remark 1, quantized all-pass fractional delay filters with the normalized lattice structure suffer from degradation in magnitude response. This fact is demonstrated in Figure 5. It is clear that the unity magnitude is violated in the normalized lattice structure, whereas the direct-form structure and the 2-multiplier lattice structure preserve the unity magnitude responses.

TABLE III
Magnitude MSEs of quantized filters with normalized lattice structure.

	$D = 8.2$	$D = 8.5$	$D = 8.8$
Normalized lattice	2.8938×10^{-4}	8.1580×10^{-5}	1.3893×10^{-4}

The quantitative magnitude error evaluation for the normalized lattice structure is shown in Table III, where the error is measured as the magnitude MSE calculated from

$$\text{MSE} = \frac{1}{M}\sum_{k=1}^{M}\left(1 - |H'_{\text{FD}}(\omega_k)|\right)^2 \tag{17}$$

with $|H'_{\text{FD}}(\omega_k)|$ denoting the magnitude response of the quantized filter evaluated at ω_k. From Fig. 5 and Table III it is intuitively concluded that the magnitude degradation in the

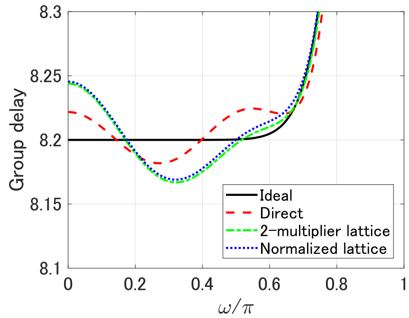

Fig. 2. Group delay characteristics of quantized all-pass fractional delay filters ($D = 8.2$).

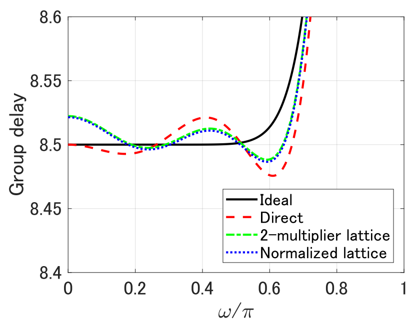

Fig. 3. Group delay characteristics of quantized all-pass fractional delay filters ($D = 8.5$).

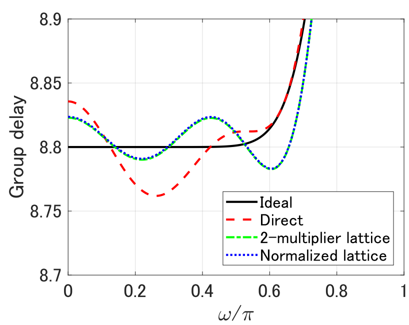

Fig. 4. Group delay characteristics of quantized all-pass fractional delay filters ($D = 8.8$).

normalized lattice structure is not a serious problem because almost flat magnitude responses are found in the low frequency region and the magnitude MSEs are small for all D.

V. Conclusion

This paper has investigated and evaluated the effect of coefficient quantization for all-pass fractional delay digital filters. It has been shown that the normalized lattice structure theoreticaly becomes the optimal structure that simultaneously minimizes the the L_1/L_2-mixed coefficient sensitivity and the L_2-coefficient sensitivity. Evalation of group delay characteristices given in Numerical examples have demonstrated this

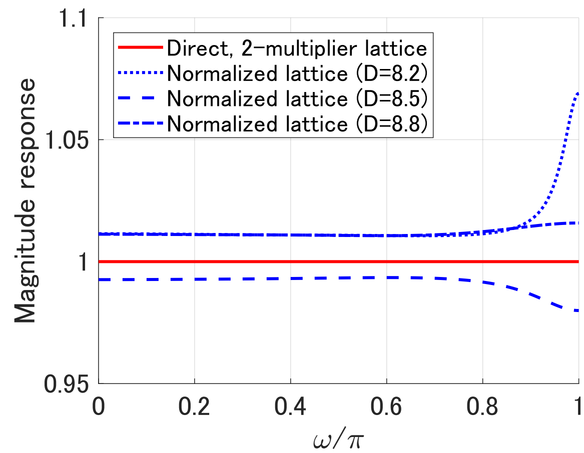

Fig. 5. Magnitude responses of quantized all-pass fractional delay filters.

theory. Also, it has been addressed that the normalized lattice structure suffers from degradation in magnitude responses because the all-pass property is violated under coefficient quantization. However this drawback is not a serious problem because the magnitude errors in the low frequency region, where the desired fractional delay are well approximated, have found to be very small.

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