Low-pass maximally flat IIR digital differentiator design with arbitrary flatness degree

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***Abstract*—Digital differentiators are often used to estimate the derivative of measured signal. They are realized as FIR (Finite Impulse Response) or IIR (Infinite Impulse Response) digital filters. Low-pass maximally flat differentiators have smooth frequency response and are useful since it can avoid signal distortion caused by ripple in the passband of the differentiator. However, the conventional design method of low-pass maximally flat IIR digital differentiator is limited for flatness degree. Hence this paper proposes a design method for low-pass maximally flat IIR digital differentiators with arbitrary flatness degree. At first, the proposed method formulates the simultaneous equations to derive the coefficients of denominator. Then, the recurrence formula is formulated to derive the coefficients of numerator. Through design examples, it is shown that the proposed method can treat arbitrary flatness degree.**

I. Introduction

Digital differentiators are widely used to estimate a time / space derivative of an input signal. In recent years, digital differentiators are paid great attention in many fields, e.g. instrumentation and control, signal processing, biomedical signal processing, and so on [1], [2], [3]. In general, input signals include noise. Since the magnitude response of full-band differentiator is similar to high-pass filter, full-band differentiator amplifies noise lying in the middle to high frequency band. To avoid this problem, low-pass filter or band-pass filter is often cascaded before or after differentiator to suppress noise. However, it is common that cascading of filter and differentiator enlarges the circuit scale [4]. Therefore, low-pass differentiator has been studied, which realize noise suppression and differentiation as one system [4], [5], [6].

A digital differentiator, including low-pass one, is usually designed as a digital filter in the frequency or z-plane. IIR (Infinite Impulse Response) digital filters [6], [14], [15], [16], [17] can realize steeper cutoff and larger attenuation than FIR (Finite Impulse Response) digital filters. Hence IIR digital filters are suitable for applications with processing time limit and circuit scale limit.

Several filter design methods minimize differential errors [14]. Such methods are efficient to suppress approximation error on whole of the passband and the stopband. However, these methods cases undesirable distortion of an input signal in the passband due to their passband ripple [10]. On the other hand, maximally flat filters [4], [5], [6], [10], [11], [15], [16] are designed to achieve that the magnitude and group delay response, and its first- to high-order derivatives are exactly the same as those of the ideal filter at $\omega = \pm\omega_0$. Thus, maximally flat filters can provide extremely high-accuracy around ω_0.

Design method for full-band maximally flat FIR digital differentiators [15], [16], low-pass ones [6] have been proposed. In general, the center frequency of low-pass maximally flat differentiator is set to $\omega_0 = 0$. Even though the method proposed in [6] can realize low-pass differentiator, the frequency resopnse with arbitrary flatness degree, which controls the bandwidth of passband in low frequency and stopband in high frequency, is not explicitly proposed. That is, users cannot adjust the bandwidth of passband or stopband of the low-pass differentiator. It is desired to derive explicit procedure for low-pass IIR digital differentiator design with arbitrary flatness degree.

This paper proposes a design method for a low-pass maximally flat IIR digital differentiator with arbitrary flatness degree. The flatness degree for stopband is controlled by the number of zeros at $\omega = \pi$. That is, the shape of numerator is limited by the stopband flatness degree. Then, the proposed method gives the frequency response of the differentiator with power series. The coefficients of denominator are derived by solving a set of linear equations. The coefficients of numerator are derived by using the coefficients of denominator. Through some design examples, the effectiveness of the proposed method is discussed.

II. The proposed method

In general, the desired frequency response of low-pass digital differentiator is given by,

$$H_d(e^{j\omega}) = \begin{cases} j\omega e^{-j\omega\tau_0}, & \omega < \omega_c \\ 0, & \omega_c < \omega \end{cases}, \tag{1}$$

where τ_0 and ω_c are the desired group delay and cutoff frequency of the differentiator. The frequency response of IIR digital differentiator is given by,

$$\begin{aligned} H(e^{j\omega}) &= \frac{B(e^{j\omega})}{A(e^{j\omega})} \\ &= \frac{\sum_{n=0}^{N} b(n)e^{-j\omega n}}{\sum_{m=0}^{M} a(m)e^{-j\omega m}}, \end{aligned} \tag{2}$$

where $a(n)$ and N are coefficients and order of numerator. In the same manner, $b(n)$ and M are coefficients and order of denominator.

When $H(e^{j\omega})$ satisfies the following flat constraints, $H(e^{j\omega})$ is called maximally flat digital differentiator.

$$\frac{d^n}{d\omega^n}|H(e^{j\omega})|\Big|_{\omega=0} = \begin{cases} 0, & n=0 \\ 1, & n=1 \\ 0, & n=2,3,\cdots,2u \end{cases} \tag{3}$$

$$\frac{d^n}{d\omega^n}\tau(e^{j\omega})\Big|_{\omega=0} = \begin{cases} \tau_0, & n=0 \\ 0, & n=1,2,\cdots,2u-1 \end{cases} \tag{4}$$

$$\frac{d^n}{d\omega^n}|H(e^{j\omega})|\Big|_{\omega=\pi} = 0, \ n=0,1,\cdots,2\nu-1 \tag{5}$$

$$2(u+\nu) = N+M. \tag{6}$$

Here $\tau(e^{j\omega})$ is the group delay of $H(e^{j\omega})$. The flatness degree, which controls the bandwidth of the passband and the stopband, are denoted by $2u$, $2u-1$ and $2\nu-1$ for $|H(e^{j\omega})|$ at $\omega=0$, $\tau(e^{j\omega})$ at $\omega=0$ and $|H(e^{j\omega})|$ at $\omega=\pi$, respectively. Note that u is a half of an odd integer and ν is a half of an integer.

From (5), $H(e^{j\omega})$ should have 2νth zeros at $\omega=\pi$. Hence, the proposed method defines numerator $A(e^{j\omega})$ as,

$$A(e^{j\omega}) = (e^{-j\omega}+1)^{2\nu}P(e^{j\omega}), \tag{7}$$

where $P(e^{j\omega})$ is N_p th order polynomial. It is obvious that $H(e^{j\omega})$ with (7) satisfies (5). In the same manner, from the case $n=0$ in (3), $H(e^{j\omega})$ should have at least one zero at $\omega=0$. Hence the order of $P(e^{j\omega})$ is $N_p \geq 1$. The order of numerator is $N=2\nu+N_p$, so that the order of denominator is $M=2u-N_p$ from (6).

To derive the other part of the frequency response, the difference between $H_d(e^{j\omega})$ and $H(e^{j\omega})$ is considered. The flatness constraints (3) and (4) can be rewritten to the flatness constraints for $H_d(e^{j\omega})-H(e^{j\omega})$ at $\omega=0$ as,

$$\frac{d^n}{d\omega^n}\left\{H_d(e^{j\omega})-H(e^{j\omega})\right\}\Big|_{\omega=0} = 0, \ n=0,1,\cdots,2u. \tag{8}$$

That is, $H_d(e^{j\omega})-H(e^{j\omega})$ should have $2u+1$th zeros at $\omega=0$,

$$H_d(e^{j\omega})-H(e^{j\omega}) = (e^{-j\omega}-1)^{2u+1}R(e^{j\omega}), \tag{9}$$

where $R(e^{j\omega})$ is an arbitrary function. By substituting (1), (2) and (7) into (9) and transforming it, the following equation holds.

$$\frac{\tilde{H}_d(e^{j\omega})}{(e^{-j\omega}-1)}A(e^{j\omega}) - P(e^{j\omega}) = (e^{-j\omega}-1)^{2u+1}\tilde{R}(e^{j\omega}) \tag{10}$$

$$\tilde{H}_d(e^{j\omega}) = \frac{j\omega e^{j\omega\tau_0}}{(e^{-j\omega}-1)^{2\nu}}, \tag{11}$$

where $\tilde{P}(e^{j\omega}) = (e^{-j\omega}+1)^{-2\nu}P(e^{j\omega})$. The power series expansion of $\tilde{H}_d(e^{j\omega})$ at $\omega=0$ is introduced as,

$$\tilde{H}_d(e^{j\omega}) = \sum_{n=0}^{\infty} d(n+1)(e^{-j\omega}-1)^{n+1} \tag{12}$$

$$d(n) = \begin{cases} 0, & n=0 \\ \displaystyle\sum_{k=1}^{n}\sum_{l=0}^{n-k}\binom{\tau_0}{l}\binom{-2\nu}{n-k-l} \cdot \frac{(-1)^k 2^{-(n-k-l)}}{k}, & n>0 \end{cases} \tag{13}$$

where $\binom{\alpha}{n}$ with a real number α and an integer n denotes,

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

Now, proposed method assumes numerator $P(e^{j\omega})$ and denominator $A(e^{j\omega})$ can be transformed as,

$$P(e^{j\omega}) = \sum_{m=0}^{N_p} p(m)(e^{-j\omega}-1)^m \tag{14}$$

$$A(e^{j\omega}) = \sum_{m=0}^{M} \tilde{a}(m)(e^{-j\omega}-1)^m. \tag{15}$$

From (12) and (15), the following equation holds.

$$\begin{aligned} \tilde{H}_d(e^{j\omega})A(e^{j\omega}) &= \left(\sum_{n=0}^{\infty} d(n+1)(e^{-j\omega}-1)^n\right) \times \left(\sum_{m=0}^{M}\tilde{a}(m)(e^{-j\omega}-1)^m\right) \\ &= \sum_{n=0}^{\infty} g(n)(e^{-j\omega}-1)^n \end{aligned} \tag{16}$$

$$g(n) = \sum_{m=0}^{\min(n,M)} \tilde{a}(m)d(n-m). \tag{17}$$

Then, by substituting (14) and (16) into (10), the following equation holds,

$$\sum_{n=0}^{\infty}\tilde{g}(n)(e^{-j\omega}-1)^n = (e^{-j\omega}-1)^{2u+1}\tilde{R}(e^{j\omega}), \tag{18}$$

where the coefficients $\tilde{g}(n)$ is given by,

$$\tilde{g}(n) = \begin{cases} g(n)-p(n), & 0 \leq n \leq N_p \\ g(n), & N_p < n \leq 2u \end{cases}. \tag{19}$$

Hence the coefficients $p(n)$ and $\tilde{a}(n)$ are derived by,

$$p(n) = g(n) = \sum_{m=0}^{\min(n,M)} \tilde{a}(m)d(n-m) \tag{20}$$

$$\tilde{a}(n) = \boldsymbol{C}^{-1}\boldsymbol{x}, \tag{21}$$

where $\boldsymbol{C}$ and $\boldsymbol{x}$ are $M-1 \times M-1$ matrix and $M-1 \times 1$ vector given by,

$$\{\boldsymbol{C}\}_{i,j} = \tilde{d}(N_p + i - j) \tag{22}$$

$$\tilde{d}(n) = \begin{cases} d(n), & n \geq 0 \\ 0, & n < 0 \end{cases} \tag{23}$$

$$\boldsymbol{x} = [\tilde{a}(0)\ 0\ 0\ \cdots\ 0]^t. \tag{24}$$

Here i, j and t denotes the row and column index of matrix, $0 \leq i \leq M-2$, $0 \leq j \leq M-2$, and vector transpose, respectively. As mentioned above, the proposed method sets $\tilde{a}(0) = 1$. With minumum numerator order $N_p = 1$, the order of denominator is $M = 2u - 1$. For this case, numerator $P(e^{j\omega})$ is employed just to realize a zero at $\omega = 0$ since $p(0) = \tilde{a}(0)d(0) = 0$. On the other hand, with maximum numerator order $N_p = 2u$, the order of denominator is $M = 0$. For this case, denominator $A(e^{j\omega})$ only gives gain $\tilde{a}(0)$, so that $H(e^{j\omega})$ can be regarded as the frequency response of the low-pass maximally flat FIR digital differentiator. By comparing the coefficients derived by [4], the proposed method sets $\tilde{a}(0) = 1$.

III. Design examples

A. Influence of τ_0

The proposed IIR differentiators are designed with same flatness degree, but different group delay. In this example, ν, u, N_p and M are set as 4, 8.5, 9 and 8, respectively. The ideal group delay at $\omega = 0$, τ_0, is varied as 5, 7, 9, 11 and 13.

Figures 1 and 2 show the magnitude response and the group delay response of the proposed low-pass differentiator. From 2, it is clear that the proposed method can realize the desired flat group delay around $\omega = 0$. From 1, it is clear that the proposed method can realize the flat magnitude response around $\omega = 0$ and $\omega = \pi$. However, there is undesired peak in the transition band as shown around $\omega = 0.7\pi$ for the example with $\tau_0 = 11$. Since such peak might amplify noise, adjustment of ideal group delay τ_0 is required to avoid peak. Figures from 3 to 7 show the zeros and poles of the proposed differentiator with varying τ_0. It is clear from these Figs. that the poles are located outside of the unit circle at first, then move into the circle with increasing value of τ_0. Hence adjustment of τ_0 is important not only for avoiding peak in the transition band, but also for the realization of causal and stable system.

TABLE I
Parameter of the proposed method in III-B.

M	ν	u	N_p	τ_0
8	5	4.5	1	2
4	5	4.5	5	10.5
0	5	4.5	9	9.5

B. Design example with different M

The proposed IIR differentiators are designed with same flatness degree, but different numerator and denominator order. In this example, ν and u are set as $\nu = 5$ and $u = 4.5$, respectively. The order of denominator M varies as 8, 4, 0. Since $u = (M+N_p)/2$ holds, N_p varies as 1, 5, 9 with respect to the value of M. The ideal group delay is adjusted to avoid peak in the transition band. The parameters of the proposed method is summarized in the table I.

Figures 8 and 9 show the magnitude response and the group delay response of the low-pass differentiator. From 8, it is clear that the proposed method can realize the flat magnitude response around $\omega = 0$ and $\omega = \pi$. It is obvious that even though each example realizes the flat passband, the cut-off characteristic of the differentiator becomes steeper with increasing value of M. From 9, it is clear that the proposed method can realize the desired flat group delay around $\omega = 0$. For the special case, $M = 0$, the proposed differentiator becomes low-pass maximally flat FIR digital differentiator. Figure 10 shows the comparison of the magnitude response of the proposed differentiator with $M = 0$ and FIR differentiator designed by [4]. It is clear that the proposed method can realize exactly same frequency response to the conventional

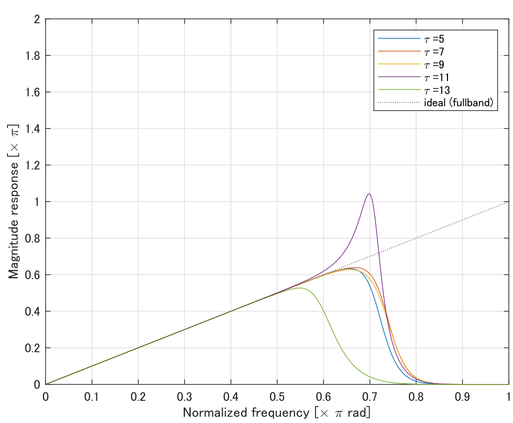

Fig. 1. The magnitude response of the proposed differentiator with different τ_0.

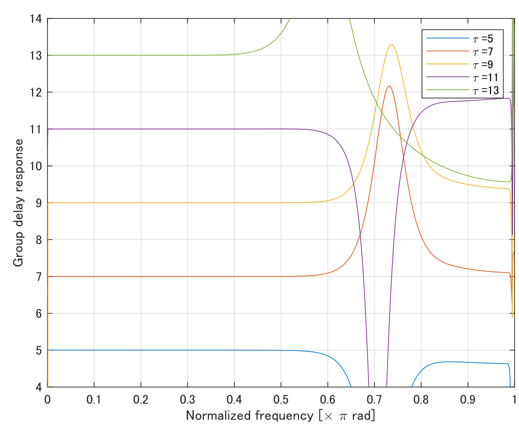

Fig. 2. The group delay response of the proposed differentiator with different τ_0.

FIR differentiator. That is, the proposed method contains the conventional design method of low-pass maximally flat FIR digital differentiator. However, the zeros and poles for all examples are not located in the unit circle as shown in from Fig. 11. Hence, these differentiators are not causal and stable. Even though the differentiator is not causal and stable, such differentiators can be used for image processing and off-line processing as mentioned in [17].

IV. Conclusions

This paper proposes a design method for a low-pass maximally flat IIR digital differentiator with arbitrary flatness degree. The proposed method gives the suitable frequency response of the differentiator. It can also express the frequency response of a low-pass maximally flat IIR digital differentiator, so that the proposed method containts the conventional design method. The coefficients of denominator are derived by solving a set of linear equations. The coefficients of numerator are derived by using the coefficients of denominator. Through some design examples, it is confirmed that the proposed method can realize the steep cut-off characteristic with same flatness degree. Hence the proposed differentiator is suitable for the derivative estimation in the noisy environment.

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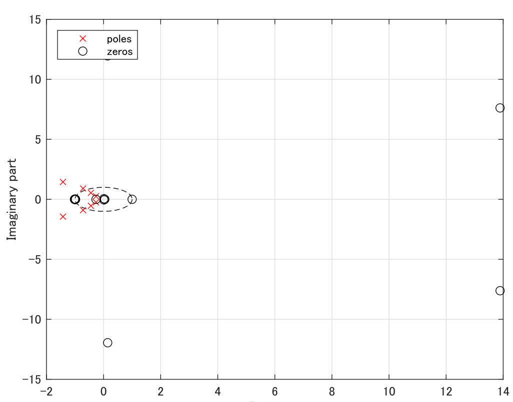

Fig. 3. The zeros and poles of the proposed differentiator with $\tau_0 = 5$.

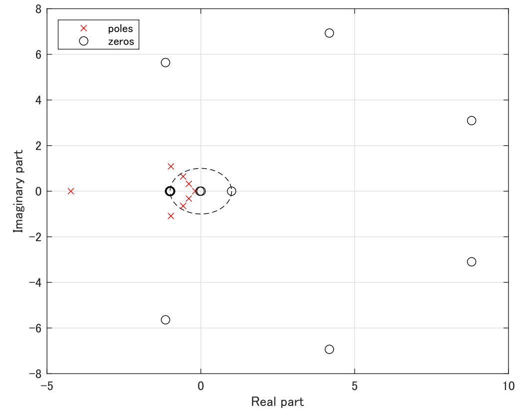

Fig. 4. The zeros and poles of the proposed differentiator with $\tau_0 = 7$.

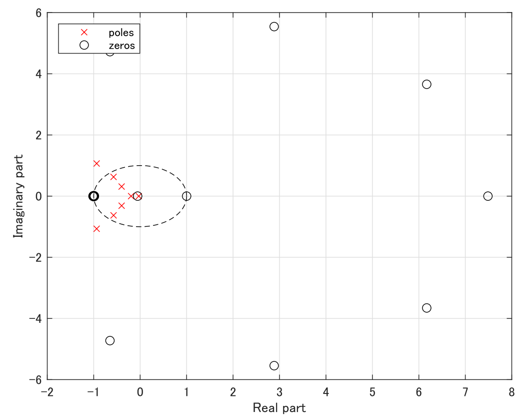

Fig. 5. The zeros and poles of the proposed differentiator with $\tau_0 = 9$.

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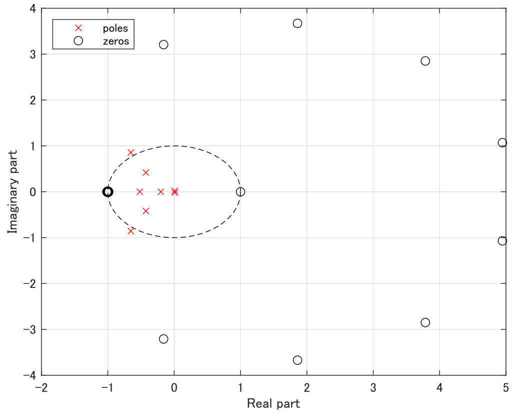

Fig. 6. The zeros and poles of the proposed differentiator with $\tau_0 = 11$.

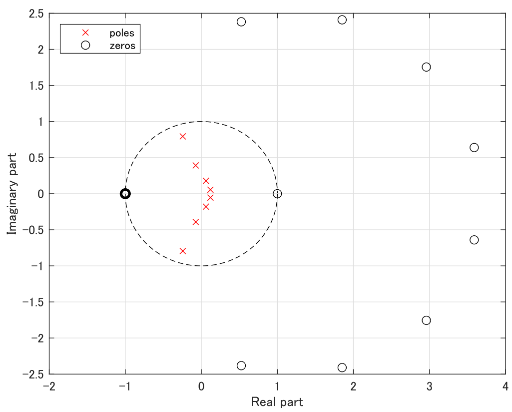

Fig. 7. The zeros and poles of the proposed differentiator with $\tau_0 = 13$.

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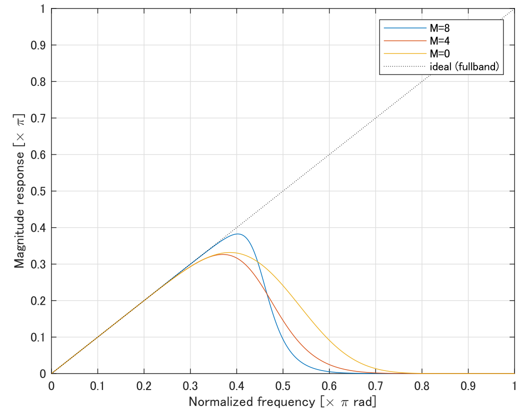

Fig. 8. The magnitude response of the proposed differentiator with $u = 4.5$ and different M.

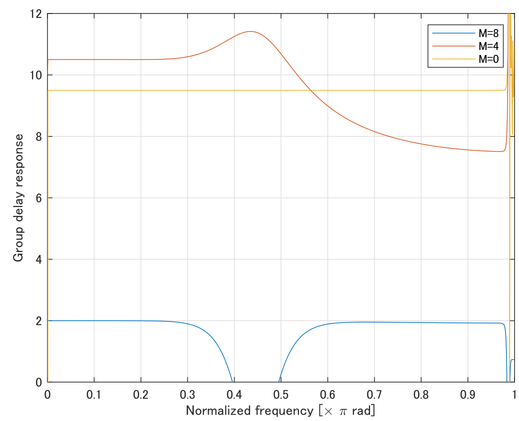

Fig. 9. The group delay response of the proposed differentiator with $u = 4.5$ and different M.

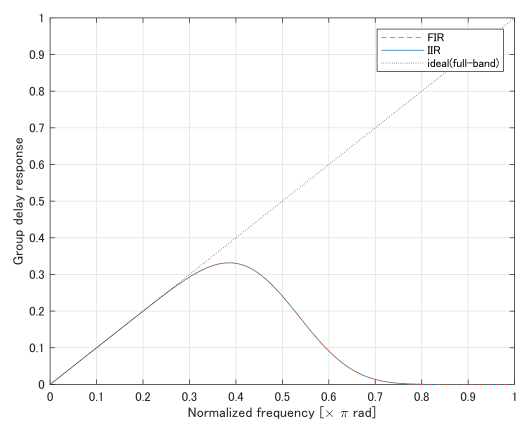

Fig. 10. The comparison of magnitude response between the proposed differentiator with $M = 0$ and the conventional low-pass maximally flat FIR digital differentiator [4].

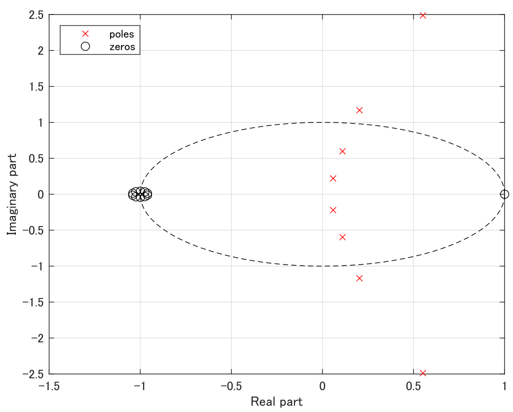

Fig. 11. Zeros and poles of the proposed differentiator with $M = 8$.