Statistical-Mechanical Analysis of Adaptive Volterra Filter for Time-Varying Unknown System

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***Abstract*—The Volterra filter is a digital filter that can describe nonlinearity. In this work, we analyze the dynamic behaviors of an adaptive signal processing system with the Volterra filter and a time-varying unknown system by a statistical-mechanical method. Specifically, assuming the self-averaging property with an infinitely long tapped-delay line, we derive simultaneous differential equations that describe the behaviors of the macroscopic variables in a deterministic and closed form and obtain the exact solution by solving them analytically. In addition, the validity of the derived theory is confirmed by comparison with numerical simulations.**

I. Introduction

Digital signal processing techniques have been used in various fields, such as information communication. In particular, the technique of changing the processing in real time in accordance with the surrounding conditions is called adaptive signal processing and is widely used. It is accomplished by updating the adaptive filter in accordance with the surrounding environment and the nature of signals.

There are several previous studies in which the behavior of adaptive filters was analyzed by the statistical-mechanical method [1]-[4]. They dealt with the case of a linear unknown system and a linear adaptive filter. However, if the unknown system is nonlinear, the adaptive filter must also have nonlinearity. One expression of nonlinearity is the Volterra series expansion. The Volterra filter uses the Volterra kernel of the Volterra series as a digital filter [5].

In this work, we describe the results of applying the statistical-mechanical method of online learning to an adaptive signal processing system consisting of a time-varying nonlinear unknown system and an adaptive Volterra filter [6]. We introduce two macroscopic variables, that is, the cross-correlation between the unknown system and the adaptive filter and the autocorrelation of the adaptive filter. Specifically, on the basis of the self-averaging when the tapped delay line is assumed to be infinitely long, we derive simultaneous differential equations of the two variables in a deterministic and closed form and obtain the exact solution by solving them analytically.

II. Volterra Filter

The Volterra filter is a digital filter that can describe nonlinearity and uses the Volterra kernel of the Volterra series expansion as the filter coefficients. The relation between input and output of Lth Volterra filter is

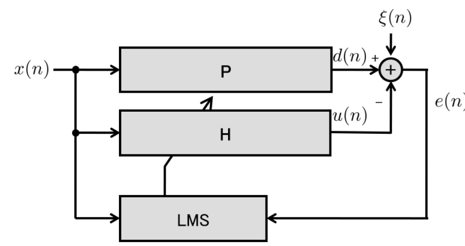

Fig. 1. Block diagram of the adaptive system.

$$\begin{aligned} y(n) &= \sum_{k_1=0}^{N-1} h_{k_1}(n)x(n-k_1) \\ &+ \sum_{k_1=0}^{N-1}\sum_{k_2=0}^{N-1} h_{k_1,k_2}(n)x(n-k_1)x(n-k_2) \\ &+ \cdots \\ &+ \sum_{k_1=0}^{N-1}\ldots\sum_{k_L=0}^{N-1} h_{k_1,k_2,\ldots,k_L}(n)\prod_{i=1}^{L} x(n-k_i). \end{aligned} \quad (1)$$

Here, $x(n)$ and $y(n)$ are the input signal and output signal of time step n, respectively. $h_{k_1,..,k_L}(n)$ is the Lth Volterra coefficient. In the adaptive Volterra filter, each Volterra coefficient $h_{k_1,..,k_L}(n)$ is updated. Volterra coefficients are symmetric, i.e., in the case of second-order coefficients,

$$h_{k_1,k_2}(n) = h_{k_2,k_1}(n) \quad (2)$$

holds [5].

III. Analytical model

The Volterra filter applied to adaptive signal processing is called the adaptive Volterra filter. Various methods used for a simple linear adaptive filter, such as the gradient method and the recursive least-squares (RLS) method, can be used to update the adaptive Volterra filter. We analyze the case in which the least mean squares (LMS) algorithm, which is one of the gradient methods, is used for updating.

Figure 1 shows a block diagram of adaptive signal processing. In Fig. 1, P and H denote the unknown system and adaptive filter, respectively. Here, $x(n)$ and $d(n)$ are the input and output of the unknown system P, $u(n)$ is the output of the adaptive filter H, $\xi(n)$ is the background noise, and $e(n)$ is the error signal. P and H are represented by a Volterra filter with tap length N and only have the second-order coefficients, $\boldsymbol{p}(n) = \{p_{k_1,k_2}(n)\}$, $\boldsymbol{h}(n) = \{h_{k_1,k_2}(n)\}$, $k_1, k_2 = 0, 1, ..., N-1$. The initial matrix $\boldsymbol{p}(0)$ is generated independently from a distribution with a mean of zero and a variance of unity. The initial matrix $\boldsymbol{h}(0)$ is set to be a zero matrix.

The unknown system has a time-varying property[7] that satisfies

$$\boldsymbol{p}(n+1) = a^{\frac{1}{N^2}}\boldsymbol{p}(n) + \sqrt{1 - a^{\frac{2}{N^2}}}\boldsymbol{w}(n), \qquad (3)$$

where each element $w_{k_1,k_2}(n)$ of $\boldsymbol{w}(n)$ is independently generated from a distribution with a mean of zero and a variance of unity at every time step. a is a parameter that controls the rate of the time variation of the unknown system. Here, $a = 1$ corresponds to the time-invariant unknown system. Note that Eq. (3) means that the norm of the coefficient vector $\boldsymbol{p}(n)$ of the unknown system P is kept constant in the mean sense, although the unknown system itself is time-varying.

The reference signal is shifted through the tapped delay line in the Volterra filter. Therefore, the tap input vector is $\boldsymbol{x}(n) = [x(n), x(n-1), \cdots, x(n-N+1)]^\top$. The input signal $x(n)$ is independently generated from a distribution with a mean of zero and a variance of $1/N$ at every time step. The outputs $d(n)$ and $u(n)$ of P and H, respectively, in time step n are

$$d(n) = \sum_{k_1=0}^{N-1}\sum_{k_2=0}^{N-1} p_{k_1,k_2}(n)x(n-k_1)x(n-k_2), \qquad (4)$$

$$u(n) = \sum_{k_1=0}^{N-1}\sum_{k_2=0}^{N-1} h_{k_1,k_2}(n)x(n-k_1)x(n-k_2). \qquad (5)$$

The error signal $e(n)$ is generated by adding an independent background noise $\xi(n)$ to the difference between $d(n)$ and $u(n)$. That is,

$$e(n) = d(n) - u(n) + \xi(n). \qquad (6)$$

Here, the background noise $\xi(n)$ is independently generated from a distribution with a mean of zero and a variance of σ_ξ^2 at every time step.

Coefficients of the adaptive Volterra filter $\boldsymbol{h}(n)$ are updated by the LMS algorithm. The LMS algorithm is a method proposed by Widrow and Hoff for minimizing the root mean square error based on the steepest descent method [8]. Therefore, the update formula is

$$\boldsymbol{h}(n+1) = \boldsymbol{h}(n) + \mu e(n)\boldsymbol{x}(n)\boldsymbol{x}(n)^\top, \qquad (7)$$

where μ is the step size parameter.

IV. Theory

In this section, we describe a theoretical analysis of the behaviors of the adaptive Volterra filter by a statistical-mechanical method. The MSE of the model used can be calculated with Eq. (6) as follows:

$$\begin{aligned}\langle e^2(n)\rangle &= \langle (d(n) - u(n) + \xi(n))^2\rangle \\ &= \langle d^2(n)\rangle + \langle u^2(n)\rangle - 2\langle d(n)y(n)\rangle + \sigma_\xi^2. \qquad (8)\end{aligned}$$

In this work, $\langle\cdot\rangle$ denotes the expectation with respect to the tap input vector $\boldsymbol{x}(n)$. Note that the background noise $\xi(n)$ is independent of the other stochastic variables and its variance σ_ξ^2 is used. Next, we focus on each term of Eq. (8). From Eq. (4), we obtain

$$\begin{aligned}\langle d^2(n)\rangle = \Bigg\langle &\sum_{k_1=0}^{N-1}\sum_{k_2=0}^{N-1}\sum_{k_1'=0}^{N-1}\sum_{k_2'=0}^{N-1} p_{k_1,k_2}(n)p_{k_1',k_2'}(n) \\ &\times x(n-k_1)x(n-k_2)x(n-k_1')x(n-k_2')\Bigg\rangle. \qquad (9)\end{aligned}$$

The right-hand side of Eq. (9) can be divided into the following four cases and others [9],[10].

$$k_1 = k_2 = k_1' = k_2' \qquad (10)$$

$$k_1 = k_1', k_2 = k_2', k_1 \neq k_2' \qquad (11)$$

$$k_1 = k_2, k_1' = k_2', k_1 \neq k_1' \qquad (12)$$

$$k_1 = k_2', k_2 = k_1', k_1 \neq k_2 \qquad (13)$$

Note that we need not consider cases other than the above four cases because their expectations are zero owing to the products of independent components. The means of the input signals corresponding to Eqs. (10)–(13) are respectively as follows:

$$\sum_{k_1=0}^{N-1}\langle x^4(n-k_1)\rangle = O\left(\frac{1}{N^2}\right), \qquad (14)$$

$$\sum_{k_1=0}^{N-1}\sum_{\substack{k_2=0\\k_2\neq k_1}}^{N-1}\langle x^2(n-k_1)x^2(n-k_2)\rangle = \frac{1}{N^2}, \qquad (15)$$

$$\sum_{k_1=0}^{N-1}\sum_{\substack{k_1'=0\\k_1'\neq k_1}}^{N-1}\langle x^2(n-k_1)x^2(n-k_1')\rangle = \frac{1}{N^2}, \qquad (16)$$

$$\sum_{k_1=0}^{N-1}\sum_{\substack{k_2=0\\k_2\neq k_1}}^{N-1}\langle x^2(n-k_1)x^2(n-k_2)\rangle = \frac{1}{N^2}. \qquad (17)$$

Now, we can rewrite Eq. (9) as

$$\begin{aligned}\langle d^2(n)\rangle =& O\left(\frac{1}{N^2}\right)\sum_{k_1=0}^{N-1} p^2_{k_1,k_1}(n)\\ &+\frac{1}{N^2}\sum_{k_1=0}^{N-1}\sum_{\substack{k_2=0\\k_2\neq k_1}}^{N-1} p^2_{k_1,k_2}(n)\\ &+\frac{1}{N^2}\sum_{k_1=0}^{N-1}\sum_{\substack{k_1'=0\\k_1'\neq k_1}}^{N-1} p_{k_1,k_1}(n)p_{k_1',k_1'}(n)\\ &+\frac{1}{N^2}\sum_{k_1=0}^{N-1}\sum_{\substack{k_2=0\\k_2\neq k_1}}^{N-1} p_{k_1,k_2}(n)p_{k_2,k_1}(n). \end{aligned}\quad (18)$$

Assuming $N \to \infty$, from Eq. (2), the second and fourth terms only remain. Therefore, we can rewrite Eq. (18) as

$$\langle d^2(n)\rangle = \frac{2}{N^2}\sum_{k_1=0}^{N-1}\sum_{\substack{k_2=0\\k_2\neq k_1}}^{N-1} p^2_{k_1,k_2}(n) = 2. \quad (19)$$

We can also obtain $\langle u^2(n)\rangle$ and $\langle d(n)u(n)\rangle$ in Eq. (8) by the same procedure as that for $\langle d^2(n)\rangle$:

$$\langle u^2(n)\rangle = \frac{2}{N^2}\sum_{k_1=0}^{N-1}\sum_{\substack{k_2=0\\k_2\neq k_1}}^{N-1} h^2_{k_1,k_2}(n), \quad (20)$$

$$\langle d(n)u(n)\rangle = \frac{2}{N^2}\sum_{k_1=0}^{N-1}\sum_{\substack{k_2=0\\k_2\neq k_1}}^{N-1} p_{k_1,k_2}(n)h_{k_1,k_2}(n). \quad (21)$$

Next, we introduce the macroscopic variables $R(n)$ and $Q(n)$ respectively defined as

$$R(n) = \frac{2}{N^2}\sum_{k_1=0}^{N-1}\sum_{k_2=0}^{N-1} p_{k_1,k_2}(n)h_{k_1,k_2}(n), \quad (22)$$

$$Q(n) = \frac{2}{N^2}\sum_{k_1=0}^{N-1}\sum_{k_2=0}^{N-1} h^2_{k_1,k_2}(n). \quad (23)$$

From Eqs. (19)–(21), we can express the MSE (8) in terms of $R(n)$ and $Q(n)$ as

$$\langle e^2(n)\rangle = 2 + 2Q(n) - 4R(n) + \sigma^2_\xi. \quad (24)$$

Now, we derive the simultaneous differential equations that describe the dynamic behaviors of the macroscopic variables $R(n)$ and $Q(n)$. From Eqs. (3) and (7),

$$p_{k_1,k_2}(n+1) = a^{\frac{1}{N^2}}p_{k_1,k_2}(n) + \sqrt{1-a^{\frac{2}{N^2}}}w_{k_1,k_2}(n), \quad (25)$$

$$h_{k_1,k_2}(n+1) = h_{k_1,k_2}(n) + \mu e(n)x(n-k_1)x(n-k_2). \quad (26)$$

By multiplying Eq. (25) by Eq. (26) and summing it over k_1 and k_2, we can obtain

$$\begin{aligned}&\sum_{k_1=0}^{N-1}\sum_{k_2=0}^{N-1} p_{k_1,k_2}(n+1)h_{k_1,k_2}(n+1)\\ &=\alpha^{\frac{1}{N^2}}\sum_{k_1=0}^{N-1}\sum_{k_2=0}^{N-1} p_{k_1,k_2}(n)h_{k_1,k_2}(n)\\ &+\mu e(n)\alpha^{\frac{1}{N^2}}\sum_{k_1=0}^{N-1}\sum_{k_2=0}^{N-1} p_{k_1,k_2}(n)x(n-k_1)x(n-k_2)\\ &+\sqrt{1-\alpha^{\frac{2}{N^2}}}w(n)\sum_{k_1=0}^{N-1}\sum_{k_2=0}^{N-1}\big(h_{k_1,k_2}(n)\\ &+\mu e(n)x(n-k_1)x(n-k_2)\big). \end{aligned}\quad (27)$$

This can be rewritten as follows using Eqs. (4) and (22):

$$\begin{aligned}&N^2R(n+1)\\ &=a^{\frac{1}{N^2}}N^2R(n)\\ &+a^{\frac{1}{N^2}}\mu e(n)d(n)\\ &+\sqrt{1-a^{\frac{2}{N^2}}}w(n)\sum_{k_1=0}^{N-1}\sum_{k_2=0}^{N-1}\big(h_{k_1,k_2}(n)\\ &+\mu e(n)x(n-k_1)x(n-k_2)\big). \end{aligned}\quad (28)$$

Note that the first terms on both sides of Eq. (28) are both $O(N^2)$ but the second term on the right-hand side is $O(1)$. Thus, to change $R(n)$ by $O(1)$, $O(N^2)$ updates are needed. Therefore, we use the value t, which is n normalized by N^2, as the time scale. By updating Eq. (28) N^2dt times in an infinitely small time dt, we can obtain N^2dt equations, as shown in Eq. (29). Note that terms that include $w(n)$ are omitted since their expectations are zero. By multiplying Eq. (29) by $a^{-\frac{1}{N^2}}, a^{-\frac{2}{N^2}}, \cdots, a^{-\frac{i}{N^2}}, \cdots, a^{-\frac{N^2dt-1}{N^2}}, a^{-\frac{N^2dt}{N^2}}$ in order from the top and summing all equations, we can obtain

$$\begin{aligned}&a^{-dt}N^2R(n+N^2dt) =\\ &\qquad N^2R(n)+\mu\left(\sum_{i=0}^{N^2dt-1} a^{-\frac{i}{N^2}}e(n+i)d(n+i)\right). \end{aligned}\quad (30)$$

Assuming $N \to \infty$, the second term on the right-hand side of Eq. (30) is replaced by its mean from self-averaging [3] as follows:

$$\begin{aligned}&N^2R(t)+N^2dR(t) =\\ &\qquad a^{dt}N^2R(t)+a^{dt}\mu\langle e(n)d(n)\rangle\sum_{i=0}^{N^2dt-1} a^{-\frac{i}{N^2}}. \end{aligned}\quad (31)$$

We define the change in $R(t)$ by updating N^2dt times as $dR(t)$. From Eq. (31), we obtain

$$\frac{dR(t)}{dt} = \left(\frac{a^{dt}-1}{dt}\right)R(t) + \frac{a^{dt}\mu}{N^2dt}\langle e(n)d(n)\rangle\sum_{i=0}^{N^2dt-1} a^{\frac{i}{N^2}}. \quad (32)$$

$$N^2dt\begin{cases} N^2R(n+1) = a^{\frac{1}{N^2}}N^2R(n) & + a^{\frac{1}{N^2}}\mu e(n)d(n) \\ N^2R(n+2) = a^{\frac{1}{N^2}}N^2R(n+1) & + a^{\frac{1}{N^2}}\mu e(n+1)d(n+1) \\ \vdots \vdots & \vdots \\ N^2R(n+N^2dt) = a^{\frac{1}{N^2}}N^2R(n+N^2dt-1) & + a^{\frac{1}{N^2}}\mu e(n+N^2dt-1)d(n+N^2dt-1) \end{cases} \quad (29)$$

Considering $dt \to 0$, we obtain

$$\frac{dR(t)}{dt} = (\ln a)R(t) + \mu\langle e(n)d(n)\rangle. \quad (33)$$

From Eqs. (19), (21), (22), and (33), the differential equation for R can be obtained as

$$\frac{dR(t)}{dt} = (\ln a - 2\mu)R(t) + 2\mu. \quad (34)$$

Next, we derive the differential equation for $Q(n)$. By squaring both sides of Eq. (26), and summing over k_1 and k_2, we can obtain

$$\begin{aligned}&\sum_{k_1=0}^{N-1}\sum_{k_2=0}^{N-1} h^2_{k_1,k_2}(n+1) \\ &= \sum_{k_1=0}^{N-1}\sum_{k_2=0}^{N-1} h^2_{k_1,k_2}(n) \\ &+ 2\mu e(n)\sum_{k_1=0}^{N-1}\sum_{k_2=0}^{N-1} h_{k_1,k_2}(n)x(n-k_1)x(n-k_2) \\ &+ \mu^2e^2(n)\sum_{k_1=0}^{N-1}\sum_{k_2=0}^{N-1} x^2(n-k_1)x^2(n-k_2). \quad (35)\end{aligned}$$

This can be rewritten as follows using Eqs. (5) and (23):

$$\begin{aligned}N^2Q(n+1) = N^2Q(n) &\\ + 2\mu e(n)u(n) &\\ + \mu^2e^2(n)\sum_{k_1=0}^{N-1}\sum_{k_2=0}^{N-1} x^2(n-k_1)x^2(n-k_2).&\end{aligned} \quad (36)$$

Similarly to the case of R, by updating N^2dt times in an infinitely small time dt and summing all equations, we can obtain

$$\begin{aligned}&N^2Q(n+N^2dt) = \\ &\quad N^2Q(n) + 2N^2dt\mu\langle e(n)u(n)\rangle + N^2dt\mu^2\langle e^2(n)\rangle.\end{aligned} \quad (37)$$

Using the value $t = n/N^2$ as the time scale and defining the change in $Q(t)$ by updating N^2dt times as $dQ(t)$, we can obtain

$$N^2dQ(t) = 2N^2dt\mu\langle e(n)u(n)\rangle + N^2dt\mu^2\langle e^2(n)\rangle. \quad (38)$$

Considering $dt \to 0$, we can obtain

$$\frac{dQ(t)}{dt} = 2\mu\langle e(n)u(n)\rangle + \mu^2\langle e^2(n)\rangle. \quad (39)$$

From Eqs. (21)–(24), the differential equation for Q can be obtained as

$$\frac{dQ(t)}{dt} = 4\mu(R(t)-Q(t)) + \mu^2\left(2 + 2Q(t) - 4R(t)) + \sigma_\xi^2\right). \quad (40)$$

The derived differential equations for $R(t)$ and $Q(t)$ (Eqs. (34) and (40), respectively) can be solved analytically, and we obtain

$$R(t) = \frac{2\mu}{2\mu - \ln\alpha}\left(1 - \alpha^t e^{-2\mu t}\right), \quad (41)$$

$$\begin{aligned}Q(t) = &\frac{8\mu^2(1-\mu)}{(2\mu-\ln\alpha)(2\mu(1-\mu)+\ln\alpha)} \\ &\times\left(-\alpha^t e^{-2\mu t} + e^{2\mu(\mu-2)t}\right) \\ &+ \frac{8\mu(1-\mu) + \mu(2+\sigma_\xi{}^2)(2\mu-\ln\alpha)}{2(2\mu-\ln\alpha)(2-\mu)} \\ &\times\left(1 - e^{2\mu(\mu-2)t}\right).\end{aligned} \quad (42)$$

By substituting these equations into Eq. (24), we can obtain the exact solution of the MSE as

$$\begin{aligned}\langle e^2(t)\rangle =& 2 + 2Q(n) - 4R(n) + \sigma^2 \\ =& -2(B-2A)\alpha^t e^{-2\mu t} + 2(B-C)e^{2\mu(\mu-2)t} \\ &+ 2 + 2C - 4A + \sigma_\xi{}^2,\end{aligned} \quad (43)$$

where A, B, and C are respectively as follows:

$$A = \frac{2\mu}{2\mu-\ln\alpha}, \quad (29)$$

$$B = \frac{8\mu^2(1-\mu)}{(2\mu-\ln\alpha)(2\mu(1-\mu)+\ln\alpha)}, \quad (30)$$

$$C = \frac{8\mu(1-\mu) + \mu(2+\sigma_\xi{}^2)(2\mu-\ln\alpha)}{2(2\mu-\ln\alpha)(2-\mu)}. \quad (31)$$

V. Results and Discussion

We compare the theoretical and simulation results. Figure 2 shows the learning curves obtained theoretically and by simulation. The conditions are $\mu = 0.1$, $\sigma_\xi^2 = 0$, and $a = 0.5, 0.7, 0.9, 1.0$. In the figure, the solid lines denote the theoretical results and the symbols denote the results of the numerical simulations. In the numerical simulations, the tap length was set to $N = 100$. Figure 2 shows that the theoretical results agree with the simulation results, so the theory explains the behavior of MSE well. In addition, it can be seen that MSE increases as the time variation of the unknown system increases. This is considered to be due to the delay in following the adaptive filter.

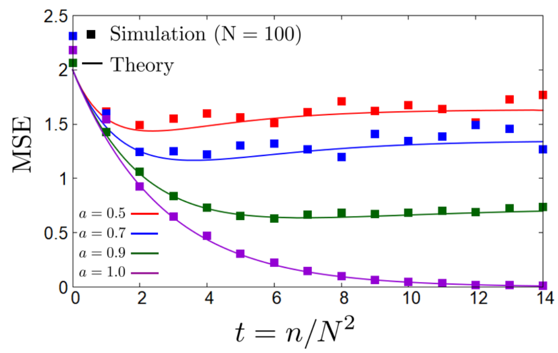

Fig. 2. Learning curves ($\sigma_\xi^2 = 0$, $\mu = 0.1$, $a = 0.5, 0.7, 0.9, 1.0$).

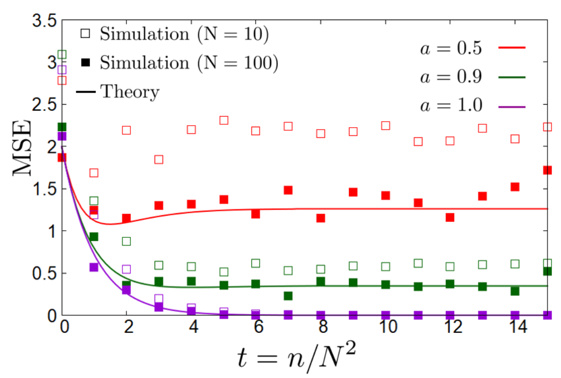

Fig. 3. Learning curves ($\sigma_\xi^2 = 0$, $\mu = 0.3$, $a = 0.5, 0.9, 1.0$).

Figure 3 shows the learning curves obtained theoretically and by simulation with varying tap length N. The conditions are $\mu = 0.3$, $\sigma_\xi^2 = 0$, and $a = 0.5, 0.9, 1.0$. In the figure, the solid lines denote the theoretical results and the symbols denote the results of the numerical simulations. In the numerical simulations, the tap length was set to $N = 10, 100$. It can be seen that when the tap length N is small, an error occurs between the numerical simulations and the theory. This is considered to be due to the finite-size effects of the tap length N. From the above results, it was confirmed that the exact solution of the MSE derived in this work is in good agreement with the results of the numerical simulations.

VI. Conclusions

In this work, we described the results of applying the statistical-mechanical method of online learning to an adaptive signal processing system consisting of a time-varying nonlinear unknown system and an adaptive Volterra filter. We introduced two macroscopic variables, that is, the cross-correlation between the unknown system and the adaptive filter and the autocorrelation of the adaptive filter. Specifically, on the basis of the self-averaging when the tapped delay line was assumed to be infinitely long, we derived simultaneous differential equations of the two variables in a deterministic and closed form and obtained the exact solution by solving them analytically. The obtained exact solution was compared with the numerical simulations, and it was found that the agreement was good even when parameter a, which controls the rate of the time variation of the unknown system, was varied. Furthermore, it was confirmed that as a increases, the adaptive filter is delayed and MSE increases, and when the tap length N is small, the finite-size effects cause errors between the theory and the numerical simulations.

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