High-accuracy reconstruction of periodic signals based on compressive sensing

Dailys Arronde Pérez [1] and Hubert Zangl [1,2]
[1] Institute of Smart Systems Technologies, Sensors and Actuators
University of Klagenfurt, Austria
[2] Silicon Austria Labs
AAU SAL USE Lab, Austria
Email: {dailys.arronde,hubert.zangl}@aau.at

***Abstract*—The successful deployment of wireless smart transducers in monitoring and control applications is conditioned by their limited power budget. However, the high impact of sampling rate on the overall power consumption of systems dealing with sparse signals can be reduced by introducing a low rate random sampling scheme based on compressive sensing. Although the reconstruction of frequency-sparse signals is a widely studied problem and solutions based on greedy and relaxation methods exist, their performance is degraded in presence of spectral leakage, which affects the sparsity of the signal representation and consequently, its estimation accuracy. In this paper a two-stage optimization approach, called Opti2, is proposed for the reconstruction of periodic signals that can be expressed in terms of fundamental frequency and harmonics. In the first stage, the estimation provided by one of the well-known sparse optimization algorithms is found and used as input to the second stage, where a constrained non-linear optimization problem is solved iteratively. The evaluation of the proposed method demonstrates that it outperforms existing approaches in terms of accuracy, showing its robustness to noise and compression rate.**

***Index Terms*—compressive sampling, spectral leakage, recovery algorithm, periodic signals**

I. Introduction

Today, many applications are in favor of wireless smart transducers, which are typically powered by batteries and energy harvesting technology with a limited capacity. In order to minimize the energy consumption in the sensor node, a more effective sampling strategy based on compressive sensing (CS) can be introduced, which reduces the sampling rate while preserving the information content of the signal when it has a sparse expansion.

Instead of acquiring N samples of the signal, in a CS system a set of $M \ll N$ measurements is generated by a linear dimensionality reduction of the form $\mathbf{y} = \mathbf{\Phi s}$, where $\mathbf{\Phi} \in \mathbb{R}^{M \times N}$ is the measurement matrix and $\mathbf{s} \in \mathbb{R}^N$ is a finite length representation of the signal. The sparse representation can be in terms of a frame or dictionary $\mathbf{\Psi}$, given by $\mathbf{s} = \mathbf{\Psi x}$, meaning that $\mathbf{x}$ has only κ out of N nonzero coefficients or it can be represented in terms of its largest κ coefficients without a significant loss. Stable recovery of sparse signals can be guaranteed from just $M = \mathcal{O}(\kappa \log(N/\kappa))$ measurements via convex optimization or iterative greedy algorithms, when the sensing matrix $\mathbf{A} = \mathbf{\Phi\Psi}$ obeys the Restricted Isometry Property (RIP) and maximal incoherence between the pairs $(\mathbf{\Phi}, \mathbf{\Psi})$ is achieved [1], [2].

In many applications, the sparse representation is in the frequency-domain and the signals are referred as spectrally sparse. Unfortunately, frequency-sparse signals are sparse with respect to the discrete Fourier transform (DFT) only if they can be expressed as a superposition of sinusoids with frequencies appearing in the lattice of those in the DFT. In practice, such signals are rarely encountered and a DFT frame of redundancy should be introduced [3], [4]. Several recovery algorithms have been developed [3]–[7] to improve the performance of the existing approaches and to extend the recovery guarantees to redundant and coherent dictionaries. The potential of ℓ_1-*analysis* for recovery of signals from undersampled data, which are sparse in a redundant dictionary, is studied in [4]. In [3], a coherence inhibition model is introduced, resulting in the sparse iterative hard thresholding (SIHT) algorithm that avoids dictionary elements with high coherence. Alternatively, algorithms based on band exclusion and local optimization techniques are proposed in [5], [6] to deal with highly coherent sensing matrices, resulting in the band-excluded local optimization orthogonal matching pursuit (BLOOMP). In [7], the band-excluded interpolating subspace pursuit (BISP) algorithm is proposed. It combines the band exclusion and polar interpolation functions in a greedy approach to improve the limitations due to the coherence and the discretization of the frequency parameter space. The polar interpolation function is based on the continuous basis pursuit (CBP) technique, proposed in [8], which is combined with orthogonal matching pursuit (OMP) in a two-stage approach to account for continuous-valued frequency estimates in [9].

Most of the previous contributions focused on the frequency estimation problem. However, some applications also require an accurate estimation of the amplitude and phase. In vibration monitoring for example, which allows to prevent equipment failures, the estimation accuracy of vibrating signal parameters is essential to correctly identify structural defects before the system reaches a critical state. In rotating machinery, the

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varying transmission path of the vibration response collected by a stationary sensor can be described as a periodic function, which helps to identify electromechanical failures related to the rotational speed of the machine [10]. Similarly, in power line measurements, the harmonic content of current or voltage waveforms should be determined at selected points of the network for power quality analysis [11]. As periodic signals are common in such applications, this paper focuses on the reconstruction of periodic signals that can be expressed in terms of its fundamental frequency and harmonics. A two-stage recovery approach to account for a high accuracy signal estimation is proposed. In the first stage, one of the existing reconstruction techniques for compressible signals is employed and used as input to the second stage, where a nonlinear optimization problem is iteratively solved to improve the estimation of the signal's parameters. Experimental results show that our approach outperforms the previously proposed algorithms for spectrally sparse signal recovery with a relatively low computational effort.

II. Compressive Sensing for periodic signals

Typically, periodic signals can be expressed in terms of the fundamental frequency and harmonics, as a superposition of K sinusoidal waveforms, with continuous time representation given by

$$s(t) = \sum_{k=1}^{K} a_k \sin\left(2\pi f_k t + \theta_k\right) \tag{1}$$

where a_k, f_k and θ_k are the amplitude, frequency and phase of each sinusoid, respectively. The fundamental frequency is given by f_1 and $f_2, f_3, \ldots, f_K$ are positive integer multiples of the fundamental frequency, then $f_k = k \cdot f_1$, for $k = 2, \ldots, K$.

Let us consider $\mathbf{s} \in \mathbb{R}^N$ a finite length discrete representation of the signal model in (1). Such signals have a κ-sparse representation in the DFT domain, with $\kappa = 2K$, only when the sinusoids have integral frequencies, i.e. they can be expressed as integer multiples of the frequency step size $\delta = f_s/N$ in the DFT basis $\mathbf{\Psi} \in \mathbb{C}^N$, for a given sampling rate f_s. Unfortunately, in the general case of non-integral frequencies, the DFT coefficients do not present the same sparsity properties due to the spectral leakage. It is worth recalling that DFT coefficients can be found by sampling the frequency-domain convolution of the discrete-time Fourier transform (DTFT) and the Dirichlet kernel, which describes the leakage effect due to the finite length sequence.

One way to overcome this problem is to introduce a redundant DFT frame or dictionary [3], [4], [6]. Such dictionary corresponds to a finer discretization of the Fourier representation, which can be seen as sampling at more closely spaced intervals. The DFT frame $\mathbf{\Psi}_p$ with redundancy factor $p \in \mathbb{N}$ contains $N_p = p \cdot N$ vectors and is defined as

$$\mathbf{\Psi}_p = \begin{bmatrix} \mathbf{e}(\omega_1) & \mathbf{e}(\omega_2) & \ldots & \mathbf{e}(\omega_{N_p}) \end{bmatrix}, \tag{2}$$

where each column vector $\mathbf{e}(\omega) \in \mathbb{C}^N$ has elements $e_n(\omega) = \frac{1}{\sqrt{N}} e^{j\omega n}, 1 \leq n \leq N$ and $\omega \in [0, 2\pi]$.

To take advantage of the sparsity property, a CS framework is employed where the signal is acquired by a reduced set of $M \ll N$ linear measurements of the form

$$\mathbf{y} = \mathbf{\Phi s} + \mathbf{n}, \tag{3}$$

where Φ is a $[M \times N]$ measurement matrix that should satisfy certain conditions [12] and $\mathbf{n}$ accounts for additive white noise in the measurement process with zero-mean and variance σ_n^2. Considering that $\mathbf{s}$ has a sparse representation $\mathbf{s} = \mathbf{\Psi}_p \mathbf{x}$ in the redundant dictionary $\mathbf{\Psi}_p$, optimal recovery of the κ-sparse signal $\mathbf{x} \in \mathbb{C}^{N_p}$ from the compressed measurements $\mathbf{y}$ is feasible, when the elements of the dictionary form an orthonormal basis, and thus are incoherent [13]. Sparse reconstruction methods can recover $\mathbf{x}$, which has a minimum number of non-zero elements (i.e. $\|\mathbf{x}\|_0 \leq \kappa$), via convex optimization or greedy algorithms. One of the most popular reconstruction techniques based on iterative greedy solutions is OMP [14] due to its low complexity and easy implementation. On the other hand, basis pursuit denoising (BPDN) [15] and linear Bregman iterations (LBI) [16] are some of the commonly used convex optimization techniques. Those are based on the ℓ_1-norm regularized optimization problem and variations, for which efficient solvers are available [17]. However, the DFT frame of redundancy in (2) violates the incoherence requirements. As the redundancy factor p increases, the κ-sparse approximation $\mathbf{x}$ becomes more accurate, but higher coherence between the frame vectors is presented. To tackle this issue, the previously mentioned algorithms for stable frequency-sparse recovery have been developed [3]–[7].

An estimation of the signal in (1) can be found from the recovery of the κ-sparse approximation $\mathbf{x}$. Even though most of the existing algorithms for sparse signal reconstruction can find a good estimation of the signal, in some applications a more refined estimation of the signal's parameters is needed, which is why a two-stage optimization approach is introduced below.

III. Opti2: Signal recovery approach

To obtain an accurate recovery of the signal $\mathbf{s}$, the parameters of each sinusoid a_k, f_k and θ_k should be estimated. Taking advantage of the periodic nature of the signal and its sparse approximation, we introduce a two-stage method that allows a precise estimation from the reduced measurements $\mathbf{y}$.

In the first stage, one of the approaches based on the relaxation of the ℓ_0-norm optimization problem is employed. Although the employed ℓ_1-norm as relaxation of the ℓ_0-norm is weaker than ℓ_0-norm in ensuring sparsity, ℓ_1-regularized optimization is a convex problem and admits efficient solution via linear programming techniques. The ℓ_1-regularized optimization is equivalent to the least absolute shrinkage and selection operator (LASSO) [18] problem, also referred to as BPDN by the signal processing community. Thus, $\mathbf{x}$ can be recovered solving the following minimization problem

$$\hat{\mathbf{x}}_1 = \min_{\mathbf{x}} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 + \lambda \|\mathbf{x}\|_1, \tag{4}$$

where $\mathbf{A} = \mathbf{\Phi}\mathbf{\Psi}_p$ is the sensing matrix and $\lambda \in [0, \infty]$ is the regularization parameter that controls the solution path. A first estimation of the signal $\hat{\mathbf{s}}_1 = \mathbf{\Psi}_p\hat{\mathbf{x}}_1$ is found and used as initial guess in the second stage. The support of $\hat{\mathbf{x}}_1$, i.e. the set of indexes associated to its κ non-zero coefficients $S_{\hat{\mathbf{x}}} = \{l : |x_l| \neq 0, l \in [1, \ldots, N_p]\}$, can be used to obtain a first estimation of the frequencies $\hat{f}_k = l_k \cdot f_s/N_p$. We exploit the fact that the frequencies $f_2, f_3, \ldots f_K$ represent the harmonics of the fundamental frequency f_1. Then we aim to refine the estimation of f_1 and the parameters a_k and θ_k by solving a non-linear optimization problem in an iterative fashion.

In the second stage, we define a search space around the estimation of $\hat{f}_1$, which is given by the interval

$$I = \left[\hat{f}_1 - c\delta, \hat{f}_1 + c\delta\right], \tag{5}$$

where δ is the frequency resolution and $c > 1$ is an integer used to control the dimension of the search space, which should contain the value of the fundamental frequency ($f_1 \in I$). Then, I is split in J subintervals of width $\Delta = \delta/2$ and the following optimization problem is solved for each subinterval.

$$\begin{aligned}(\hat{\mathbf{a}}_j, \hat{f}_j, \hat{\boldsymbol{\theta}}_j) &= G(\mathbf{y}, \mathbf{\Phi}, f_{c_j}) \\ &= \arg\min_{\mathbf{a}, f, \boldsymbol{\theta}} \|\mathbf{y} - \mathbf{\Phi}\tilde{\mathbf{s}}\|_2^2 \\ \text{s.t.} &\left\{\begin{matrix} a_k \geq 0, \\ f_{c_j} - \frac{\Delta}{2} \leq f \leq f_{c_j} + \frac{\Delta}{2}, \\ 0 \leq \theta_k \leq \pi, \end{matrix}\right\} \text{for } k = 1, \ldots, K\end{aligned} \tag{6}$$

where $\tilde{\mathbf{s}} = [\tilde{s}_1\ \tilde{s}_2\ \ldots\ \tilde{s}_N]$ is given by

$$\tilde{s}_n = \sum_{k=1}^{K} a_k \sin(2\pi k f n + \theta_k), n \in \{1, 2, \ldots N\}, \tag{7}$$

f_{c_j} is the central value of the j-th interval and the vectors $\mathbf{a} = [a_1\ a_2\ \ldots\ a_K]$ and $\boldsymbol{\theta} = [\theta_1\ \theta_2\ \ldots\ \theta_K]$ comprise the amplitudes and phases of the sinusoidal waveforms. The set of parameters which result in the minimum squared error is then selected and used for estimating $\hat{\mathbf{s}}$. The number of sinusoids composing the signal $\mathbf{s}$ is generally unknown, then K in (7) is assumed to be the order of the highest harmonic component that can be detected for the specified sampling rate, i.e. $K = \left\lceil \frac{f_s}{2f_1} \right\rceil$. For instance, if a signal with the fundamental frequency $f_1 = 1\,\text{kHz}$ and the sampling rate $f_s = 10\,\text{kHz}$ are considered, then K is set to 5. The proposed approach is summarized in Algorithm 1.

It is worthy to note that in the first stage it is possible to use either a greedy iterative algorithm or convex optimization approach. The performance of the proposed approach is determined by the accuracy of the first estimation. If the first estimation of the fundamental frequency fails, the second stage will also provide a poor result. Nevertheless, studies have shown that the existing sparse recovery techniques provide a reliable reconstruction of the signal.

IV. Simulation Results

The performance of the proposed approach is evaluated through a set of numerical experiments that have been carried out in Matlab programming environment. The performance is measured in terms of the mean squared error (MSE) of the estimated signal $\hat{\mathbf{s}}$ via Monte Carlo (MC) experiments and averaged over $n_{MC} = 30$ independent trials.

Algorithm 1: Opti2

Input: Compressed measurements $\mathbf{y}$, measurement matrix $\mathbf{\Phi}$, redundant frame or dictionary $\mathbf{\Psi}_p$ and frequency resolution δ

Output: Reconstructed signal $\hat{\mathbf{s}}$

- $\mathbf{A} = \mathbf{\Phi}\mathbf{\Psi}_p$, $\Delta = \delta/2$
- Find $\hat{\mathbf{x}}_1$ using (4) $\rightarrow \hat{\mathbf{s}}_1 = \mathbf{\Psi}_p\hat{\mathbf{x}}_1,\ \hat{f}_1 = l_1 \cdot f_s/N_p$
- $I = [f_{I_{min}}, f_{I_{max}}]$ where $f_{I_{min}} = \hat{f}_1 - c\delta,\ f_{I_{max}} = \hat{f}_1 + c\delta$
- Split I in j subintervals $I = \bigcup_{j=1}^{J} [f_{j_{min}}, f_{j_{max}})$ where $f_{j_{min}} = f_{I_{min}} + (j-1)\Delta, f_{j_{max}} = f_{I_{min}} + j\Delta$
- **for** $j = 1$ **to** J **do**
  - $f_{c_j} = (f_{j_{min}} + f_{j_{max}})/2$
  - $(\hat{\mathbf{a}}_j, \hat{f}_j, \hat{\boldsymbol{\theta}}_j) = G(\mathbf{y}, \mathbf{\Phi}, f_{c_j})$ using (6)
  - error$(j) = \|y - \Phi\hat{\mathbf{s}}_j\|_2^2$
- **end**
- $j_{min} = \arg\min_j$ error
- $(\hat{\mathbf{a}}, \hat{f}, \hat{\boldsymbol{\theta}}) = (\hat{\mathbf{a}}_{j_{min}}, \hat{f}_{j_{min}}, \hat{\boldsymbol{\theta}}_{j_{min}})$
- $\hat{\mathbf{s}} = \sum_{k=1}^{K} \hat{a}_k \sin(2\pi k \hat{f} n + \hat{\theta}_k)$

$$MSE = \frac{1}{n_{MC}} \sum_{i=1}^{n_{MC}} \frac{1}{N} \|\hat{\mathbf{s}} - \mathbf{s}\|_2^2. \tag{8}$$

An observation interval of $30\,\text{ms}$ and a sampling rate $f_s = 10\,\text{kHz}$ are considered for the simulated data, then discrete signals of length $N = 300$ containing $K = 4$ sinusoidal waveforms are generated. The amplitudes and fundamental frequency are selected uniformly at random at each experiment, while the phases are assumed to be in the interval $[0, \pi]$. A DFT frame with redundancy factor $p = 5$ is considered and the measurement matrix $\mathbf{\Phi} \in \mathbb{R}^{M \times N}$ is assumed to be Gaussian, which guarantees that the sensing matrix fulfills the RIP [19]. The reduced set of measurements $\mathbf{y}$ from which the signal of interest is recovered, is generated using (3).

Opti2 is compared with state-of-the-art methods to reconstruct sparse signals: BPDN, LBI, l_1-analysis, OMP+CBP, BLOOMP and BISP. The evaluated approaches aim to recover the sparse representation $\hat{\mathbf{x}}$, from which the signal is obtained by $\hat{\mathbf{s}} = \mathbf{\Psi}_p\hat{\mathbf{x}}$, except l_1-analysis which directly finds the estimation $\hat{\mathbf{s}}$. To solve the non-linear optimization problem in (6), the built-in Matlab function *fmincon* was used, where f_{c_j}, $\hat{a}_k = \max|\hat{\mathbf{s}}_1|$ and $\hat{\theta}_k = 0$, for $k = 1, \ldots, K$ were used as initial guess. The toolbox [20] was employed for solving (4) and [21], for the convex problems in l_1-analysis and CBP. On the other hand, the codes for the implementation of LBI, BLOOMP and BISP are available in [22], [23] and [24], respectively.

Fig. 1 depicts the results of the first experiment, where a fixed number of measurements $M = 150$ and varying signal-

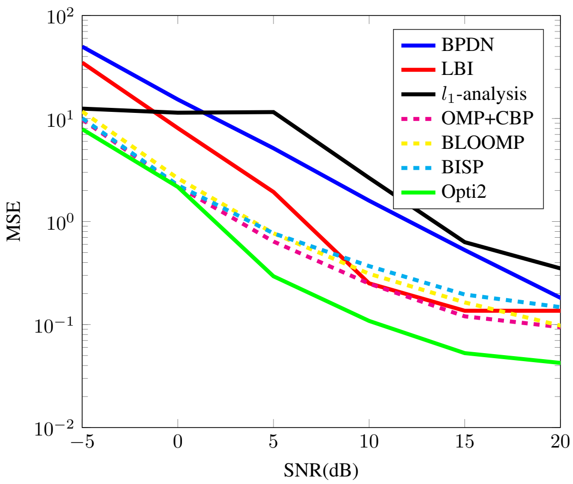

Fig. 1. Performance comparison of different reconstruction algorithms in terms of the MSE vs. SNR.

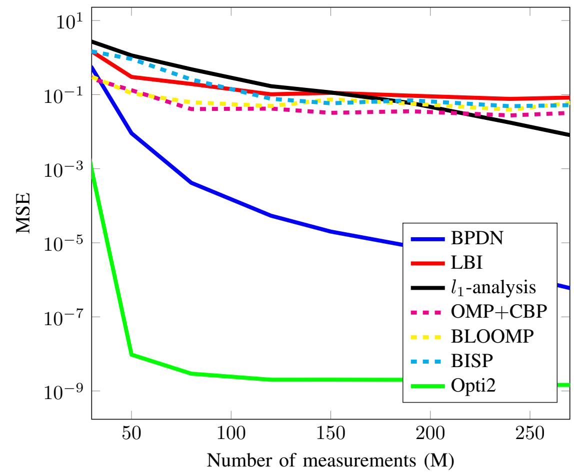

Fig. 2. Performance comparison of different reconstruction algorithms in terms of the MSE vs. M.

to-noise ratio (SNR) from -5 to $20\,\mathrm{dB}$ are considered. Noisy observations from the measurement model described in (3) are simulated, where the SNR is defined as the ratio between the power of the compressed noiseless measurements and the noise variance σ_n^2. The solid lines depict algorithms based on convex optimization while the dashed lines depict the ones based on greedy approaches. It can be observed that for low SNR values, BPDN shows the worst performance followed by LBI and l_1-analysis. However, for higher SNR values l_1-analysis presents the worst performance. On the other hand, BLOOMP, BISP and OMP+CBP have comparable performance, where the latter slightly outperforms the other two. Note that Opti2 outperforms all the evaluated approaches for the considered SNR region, reducing the MSE of the estimated signal in the second optimization step and overcoming the effects caused by the spectral leakage.

To analyse the accuracy of the estimated parameters $\hat{\mathbf{a}}, \hat{f}$ and $\hat{\boldsymbol{\theta}}$, the MSE is evaluated after the first and second stages. The vectors $\hat{\mathbf{a}}$ and $\hat{\boldsymbol{\theta}}$ comprise the estimations of the amplitudes $\hat{a}_k$ and phases $\hat{\theta}_k$ corresponding to each sinusoid in (7), while $\hat{f}$ provides the estimation of the fundamental frequency. The set $S_{\hat{\mathbf{x}}}$ is used to obtain the estimation of the parameters in the first stage. An initial estimation of the fundamental frequency is found by $\hat{f}_1 = l_1 \cdot f_s/N_p$, with $l_1 \in S_{\hat{\mathbf{x}}}$ being the index associated with the highest non-zero coefficient. First estimates of amplitude and phase are computed as $\hat{\mathbf{a}}_1 = |\hat{\mathbf{x}}_S|$ and $\hat{\theta}_1 = \arg(\hat{\mathbf{x}}_S)$ respectively, where the coefficients $\hat{\mathbf{x}}_S$ are determined by the least-squares solution

$$\hat{\mathbf{x}}_S = \frac{1}{N_p}\left(\boldsymbol{\Psi}_{p_S}^{H}\boldsymbol{\Psi}_{p_S}\right)^{-1}\boldsymbol{\Psi}_{p_S}^{H}\boldsymbol{\Phi}^{T}\mathbf{y}, \tag{9}$$

with $\boldsymbol{\Psi}_{p_S}$ been obtained by keeping the columns of $\boldsymbol{\Psi}_p$ indexed by $S_{\hat{\mathbf{x}}}$. The estimation of the individual parameters in the second stage are directly obtained from (6). The results in Table I involve 100 independent experiments for noisy observations with $20\,\mathrm{dB}$ of SNR. It can be noticed that the second optimization stage significantly improves the estimations obtained in the first stage, for both parameters and signal.

TABLE I
MSE OF THE ESTIMATED PARAMETERS AND THE ESTIMATED SIGNAL AT EACH STAGE OF THE PROPOSED APPROACH, CONSIDERING SNR $= 20\,\mathrm{dB}$

	MSE			
	Amplitude ($\hat{\mathbf{a}}$)	Frequency ($\hat{f}$)	Phase ($\hat{\boldsymbol{\theta}}$)	Signal ($\hat{\mathbf{s}}$)
BPDN	0.1280	4.1163	1.1038	0.1253
Opti2	0.0025	0.4844	0.0928	0.0356

For the second experiment, the MSE is evaluated in terms of the number of measurements M. We set $M = \beta N$, where $\beta \in (0, 1]$ and a range of subsampling ratios β is explored with noiseless measurements to verify the compression level that allows a successful estimation. The results are shown in Fig. 2. Opti2 achieves the best performance, converging to a minimum MSE with about 20% of compression rate. BPDN outperforms the other approaches, improving its estimation accuracy as the number of measurements increases. For a small number of measurements, l_1-analysis presents the worst behavior. LBI and the greedy approaches present a comparable performance, which do not improve considerably with an increasing number of measurements.

The average computation time of the evaluated approaches is listed in Table II, considering a scenario with SNR $= 20\,\mathrm{dB}$ and $M = 150$ measurements. The proposed approach is faster than BISP and l_1-analysis, which becomes extremely slow for higher dimensions (N, M). It should be taken into account that the computation time of Opti2 includes the delay of the first optimization step (in this case BPDN). Consequently, if the processing time should be reduced, a faster approach as LBI or OMP can be employed to get the first estimation of the signal. The second optimization step carry out in Opti2 does not increase the computation time considerably. Note that the computations were performed on a notebook featuring an Intel i7-8550U 4 Core processor.

TABLE II
AVERAGE RUNTIME IN SECONDS

Algorithm	Time (s)
BPDN	4.9796
LBI	0.5075
l_1-analysis	183.0004
OMP+CBP	1.3960
BLOOMP	0.1332
BISP	12.1184
Opti2	6.5188

V. CONCLUSIONS

In this paper, we propose a two-stage reconstruction approach, referred as Opti2, to improve the estimation of periodic signals that can be expressed as a sum of real-valued sinusoids. The estimation provided by one of the well-established recovery techniques for compressed measurements is used as first estimation, which gets refined in a second stage by iteratively solving a non-linear optimization problem. Experimental results show that Opti2 outperforms reported techniques used in spectral compressive sensing, achieving accurate results in terms of the MSE.

REFERENCES

[1] E. J. Candes and M. B. Wakin, "An introduction to compressive sampling," *IEEE Signal Processing Magazine*, vol. 25, no. 2, pp. 21–30, 2008.

[2] R. G. Baraniuk, "Compressive sensing [lecture notes]," *IEEE Signal Processing Magazine*, vol. 24, no. 4, pp. 118–121, 2007.

[3] M. F. Duarte and R. G. Baraniuk, "Spectral compressive sensing," *Applied and Computational Harmonic Analysis*, vol. 35, no. 1, pp. 111 – 129, 2013. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S1063520312001315

[4] E. J. Candes, Y. C. Eldar, D. Needell, and P. Randall, "Compressed sensing with coherent and redundant dictionaries," *Applied and Computational Harmonic Analysis*, vol. 31, no. 1, pp. 59 – 73, 2011. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S1063520310001156

[5] A. Fannjiang and W. Liao, "Coherence-pattern guided compressive sensing with unresolved grids," *CoRR*, vol. abs/1106.5177, 06 2011.

[6] A. Fannjiang and W. Liao, "Super-resolution by compressive sensing algorithms," in *2012 Conference Record of the Forty Sixth Asilomar Conference on Signals, Systems and Computers (ASILOMAR)*, 2012, pp. 411–415.

[7] K. Fyhn, H. Dadkhahi, and M. F. Duarte, "Spectral compressive sensing with polar interpolation," in *2013 IEEE International Conference on Acoustics, Speech and Signal Processing*, 2013, pp. 6225–6229.

[8] C. Ekanadham, D. Tranchina, and E. P. Simoncelli, "Recovery of sparse translation-invariant signals with continuous basis pursuit," *IEEE Transactions on Signal Processing*, vol. 59, no. 10, pp. 4735–4744, 2011.

[9] M. Bertocco, G. Frigo, and C. Narduzzi, "High-accuracy frequency estimation in compressive sensing-plus-dft spectral analysis," in *2015 IEEE International Instrumentation and Measurement Technology Conference (I2MTC) Proceedings*, 2015, pp. 1668–1671.

[10] G. He, K. Ding, W. Li, and Y. Li, "Frequency response model and mechanism for wind turbine planetary gear train vibration analysis," *IET Renewable Power Generation*, vol. 11, no. 4, pp. 425–432, 2017. [Online]. Available: https://ietresearch.onlinelibrary.wiley.com/doi/abs/10.1049/iet-rpg.2016.0236

[11] D. F. Fabri, C. H. N. Martins, L. R. M. Silva, C. A. Duque, P. F. Ribeiro, and A. S. Cerqueira, "Time-varying harmonic analyzer prototype," in *Proceedings of 14th International Conference on Harmonics and Quality of Power - ICHQP 2010*, 2010, pp. 1–7.

[12] E. Candes, J. Romberg, and T. Tao, "Stable signal recovery from incomplete and inaccurate measurements," *Communications on Pure and Applied Mathematics*, vol. 59, 08 2006.

[13] E. Candes and J. Romberg, "Sparsity and incoherence in compressive sampling," *Inverse Problems*, vol. 23, 11 2006.

[14] J. A. Tropp and A. C. Gilbert, "Signal recovery from random measurements via orthogonal matching pursuit," *IEEE Transactions on Information Theory*, vol. 53, no. 12, pp. 4655–4666, 2007.

[15] S. S. Chen, D. L. Donoho, and M. A. Saunders, "Atomic decomposition by basis pursuit," *SIAM Review*, vol. 43, no. 1, pp. 129–159, 2001. [Online]. Available: https://doi.org/10.1137/S003614450037906X

[16] J.-F. Cai, S. Osher, and Z. Shen, "Linearized bregman iterations for compressed sensing," *Math. Comput.*, vol. 78, pp. 1515–1536, 09 2009.

[17] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, March 2004. [Online]. Available: http://www.amazon.com/exec/obidos/redirect?tag=citeulike-20\&path=ASIN/0521833787

[18] T. Hastie, R. Tibshirani, and M. Wainwright, *Statistical Learning with Sparsity: The Lasso and Generalizations*. Chapman and Hall,CRC, 2015.

[19] R. Baraniuk, M. Davenport, R. DeVore, and M. Wakin, "A simple proof of the restricted isometry property for random matrices," *Constructive Approximation*, vol. 28, pp. 253–263, 12 2008.

[20] M. P. Friedlander and E. van den Berg, https://www.cs.ubc.ca/~mpf/spgl1/index.html.

[21] M. Grant and S. Boyd. (2013, Sep.) Cvx: Matlab software for disciplined convex programming, version 2.0 beta. . [Online]. Available: http://cvxr.com/cvx

[22] Z. Yin and Y. Wotao, https://www.caam.rice.edu/~optimization/L1/bregman/.

[23] W. Liao, http://people.math.gatech.edu/~wliao60/.

[24] M. F. Duarte, http://www.ecs.umass.edu/~mduarte/Software.html.