Feedback Quantization and Bit Allocation for Networked Control Systems with Rate Limited Channels

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***Abstract*—This paper considers feedback quantization and bit allocation for networked control where observation signals are transmitted through a wireless digital communication channel. Since the channel can transmit only a limited number of bits per unit time, bits have to be allocated to each observation signal efficiently. To obtain reasonable performance at a relatively low cost, we employ feedback quantizers to quantize continuous signals. First, we express the mean squared error (MSE) of the output of a networked control system due to quantization in terms of allocated bits and feedback filters of quantizers. Then, since it is hard to obtain the optimal bit allocation and feedback filters that jointly minimize MSE, we propose a heuristic design that iteratively optimizes bit allocation and feedback filters. The bits are allocated by a greedy algorithm, whereas feedback filters are designed by solving optimization problems. A numerical example shows that our design exhibits better performance than the conventional design.**

I. Introduction

Wireless communication technology plays a greater role also in networked control systems, where multiple sensors at a controlled plant send their observation signals to a controller and the controller transmits control signals to actuators at the plant through communication channels (e.g. see [1], [2] and the references therein). However, there still remain some critical constraints on wireless communications, e.g. bit rate limitations of communication channels, communication delays, data packet losses, bit errors, asynchronicity, and so on.

Although such constraints heavily affect the performance of the networked control system, there have not been many works on control taking account of the communication constraints. Most of control laws have been developed under the ideal assumption that the communication is perfect. For discrete-time, linear and time-invariant (LTI) systems, stabilizability over a rate limited communication channel has been studied by using ideal quantization [3], [4]. On the other hand, some of communication technologies have been developed without paying much attention to applications over them. For example, channel coding often treats each bit in a bit stream equally, although bits may have different effects on the system performance. A better performance can be obtained if one designs the system considering the control and the communication simultaneously. Indeed, a simple on-off error control coding can improve the control quality [5].

In [6], joint optimization of wireless communication and networked control has been considered by using conventional uniform quantizers. A cross layer design for communication and control of a networked control system has been presented in [7] to achieve a better control accuracy with uniform quantizers. In this paper, a cross layer design of quantization for networked control over a rate limited communication channel is presented.

When sensors transmit their observation signals to the controller independently, we have to allocate the fixed bit rate to communication links. TDMA-based allocation has been proposed for sensor networks [8] and for a networked control system having only one observation signal [9]. On the other hand, to attain asymptotic stability of a networked control system, quantization which adaptively changes its quatization step-size has been provided in [10]. However, it works only when the signal to be quantized converges to a certain value and the limiting value is known *a priori*.

We design a feedback quantizer, which is composed of an error feedback filter and a uniform quantizer (see Fig. 2 in Sec. II), for a networked control system over a rate limited channel. First, we evaluate the mean squared error (MSE) in the output of the networked control system due to quantization. MSE is expressed by assigned bits and feedback filters. Then, since it is hard to obtain the optimal bit allocation and feedback filters that jointly minimize MSE, we propose a heuristic design that iteratively optimizes bit allocation and feedback filters. The bits are allocated by a greedy algorithm [12], whereas the feedback filters are designed by solving optimization problems. A numerical example shows that our design exhibits better performance than the conventional design.

II. Networked control and quantization

Fig. 1 shows a control system, where the plant is a single input and single output linear time-invariant (LTI) system. For simplicity, we consider the discretized system with discrete-time signals. The discretized plant is assumed to be reachable and observable.

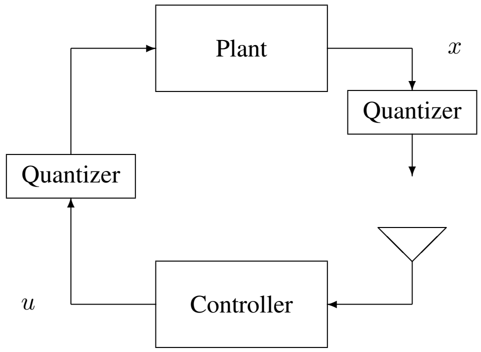

Fig. 1. Control system and quantization.

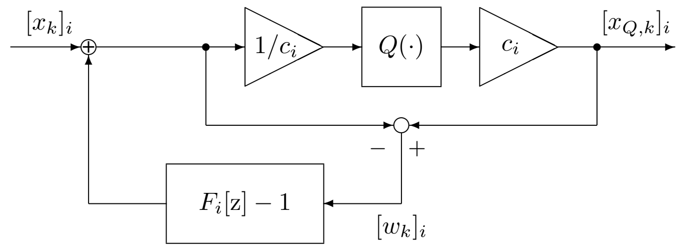

Fig. 2. Feedback quantizer.

This paper only considers the quantization for the communication from the plant to the controller. A similar result may be obtained for the communication from the controller to the plant.

At time k, the controller generates a control signal u_k based on the observation signals x_k from the plant. For simplicity of presentation, we assume that the $n \times 1$ state vector of the plant is directly observed and x_k is the state vector of the plant at time k.

The state vector x_k is transmitted over a wireless communication channel. Without loss of generality, we assume that the ranges of the signals are symmetric such that $|[x_k]_i| \leq L_i$ for any $k \in [1, n]$ where $[x_k]_i$ is the ith entry of x_k.

We quantize the entries of x_k independently and adopt feedback quantizers for quantization, since they exhibit better performance than conventional uniform quantizers at the expense of a relatively low cost. Feedback quantizers are equivalent to linearized $\Delta\Sigma$ modulators [11], which are often utilized for analog to digital (A/D) or digital to analog (D/A) conversions.

Fig. 2 illustrates a feedback quantizer for the ith entry of x_k, where $Q(\cdot)$ is a uniform quantizer whose saturation levels are ± 1, $[x_k]_i$ is the ith entry of x_k, $[x_{Q,k}]_i$ is the output of ith quantizer, and $[w_k]_i$ is the quantization error of the uniform quantizer $Q(\cdot)$.

To avoid the overloading, the input to the uniform quantizer is scaled by the factor $1/c_i$ for $c_i > 0$. The output of the uniform quantizer is re-scaled by the factor of c_i.

The round-off error of the uniform quantizer is filtered by an error feedback filter $F_i[\mathrm{z}]-1$ and is fed back to the input. The feedback filter $F_i[\mathrm{z}]-1$ is strictly proper, that is, $F_i[\infty] = 1$. It should be remarked that when $F_i[\mathrm{z}] = 1$, there is no feedback and the quantizer becomes a uniform quantizer.

We assign b_i bits to the ith entry of x_k. Then, the required bit rate B is given by

$$B = \sum_{i=1}^{n} b_i. \tag{1}$$

On the other hand, the quantization interval d_i for the uniform quantizer is given by

$$d_i = \frac{1}{2^{b_i - 1}}. \tag{2}$$

We assume that the quantization error of a uniform quantizer is uniformly distributed white random noise with zero mean. Then, the variance of the round-off error is given by

$$\sigma_i^2 = \frac{1}{3}\left(\frac{d_i}{2}\right)^2 = \frac{1}{3 \cdot 2^{2b_i}}. \tag{3}$$

In general, this whiteness assumption on the quantization error does not hold true for feedback quantizers unless there is no overflow, the quantization interval is sufficiently small and the number b_i is sufficiently large. However, we adopt it, since it can simplify the design of bit allocation and feedback quantizers to obtain better performance than the conventional uniform quantizers.

Let us denote the L_2 norm of $G[\mathrm{z}]$ as $\|G[\mathrm{z}]\|$, which is defined as

$$\|G[\mathrm{z}]\| = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} G^*[e^{j\omega}]G[e^{j\omega}]d\omega\right)^{\frac{1}{2}} \tag{4}$$

where c^* is the complex conjugate of c. If the impulse response of the feedback filter $F_i[\mathrm{z}]-1$ is not sparse, the output of the feedback filter may be approximated as a Gaussian random variable whose variance is given by

$$\sigma_{\eta_i}^2 := \|F_i[\mathrm{z}] - 1\|^2 c_i^2 \sigma_i^2. \tag{5}$$

To avoid overflow (or equivalently overloading), we have to regulate the feedback signal. Suppose that the probability of overflow should be less than $1-P$ for $0 < P \leq 1$. If the feedback signal is Gaussian, we can find a value c_P so that the probability of the absolute value of the feedback signal being less than $c_P\sigma_{\eta_i}$ is P. Then, if we put

$$c_i = L_i + c_P\sigma_{\eta_i} \tag{6}$$

the probability of overflows is less than $1-P$. From (5) and (6), we have

$$c_i = \frac{1}{1 - c_P\|F_i[\mathrm{z}]-1\|\sigma_i} L_i \tag{7}$$

which necessitates $\|F_i[z] - 1\| \leq 1/(c_P\sigma_i)$.

We have to use the quantized state vector $x_{Q,k}$ in place of x_k. Let us employ a state feedback control and generate the control signal by

$$u_k = K(r_k - x_{Q,k}) \tag{8}$$

where r_k is a reference for the state vector x_k and $K = [k_1, k_2, ..., k_n]$ is the feedback gain which is determined by a control law, e.g. an optimal control.

The input to the plant can be expressed as $u_k = K(r_k - x_k) + Ke_k$, where $e_k := x_{Q,k} - x_k$ is the quantization error of the feedback quantizer at time k. It is noted that the term $K(r_k - x_k)$ is the control signal when there is no error, while the term Ke_k is due to quantization errors.

Let the discrete-time transfer function from the input u_k of the plant to the output y_k of the plant be $G[z]$. Then, the z transform of y_k can be expressed as

$$Y[z] = G[z]K(R[z] - X[z]) + G[z]\sum_{i=1}^{n} k_i F_i[z]W_i[z] \tag{9}$$

where $R[z]$, $X[z]$, and $W_i[z]$ are z transforms of r_k, x_k, and $[w_k]_i$, respectively.

Assuming that each entry of the quantization error w of the uniform quantizer is not correlated with each other, the mean squared error (MSE) due to the quantization is given by

$$\text{MSE} = \sum_{i=1}^{n} \|G[z]F_i[z]\|^2 k_i^2 c_i^2 \sigma_i^2. \tag{10}$$

Substituting (2) and (7) into (10), we finally have

$$\text{MSE} = \sum_{i=1}^{n} \frac{\|G[z]F_i[z]\|^2 k_i^2 L_i^2}{\left(1 - \frac{c_P\|F_i[z]-1\|}{\sqrt{3}2^{b_i}}\right)^2} \frac{1}{3 \cdot 2^{2b_i}} \tag{11}$$

Our objective is to find bit allocation and feedback filters that minimize MSE when the total number of bits is fixed as in (1), for given system $G[z]$, feedback gain K and input range L_i for $i = 1, \ldots, n$. Unlike [6] that considers the allocation of communication resources for uniform quantizers, we design feedback filters to obtain better performance.

III. A Heuristic Design

In general, MSE is not a convex function in our design variables. To obtain reasonable bit allocation and feedback filters, we propose a heuristic method that iterates the bit allocation and the design of feedback filters.

First, we start the uniform quantization, that is, the case where $F_i[z] = 1$ for all $i \in [1, n]$ in (11). The MSE can be expressed as

$$\text{MSE} = \frac{\|G[z]\|^2}{3} \sum_{i=1}^{n} \frac{k_i^2 L_i^2}{2^{2b_i}}. \tag{12}$$

In this case, the MSE is a sum of convex and strictly decreasing functions in b_i. Thus, the optimal bit allocation can be obtained by a greedy algorithm [13].

We assign one bit at a time to one of the quantizers that reduces the MSE most. We repeat this until all B bits are assigned. The greedy algorithm for our bit allocation can be summarized in the following steps:

1) Allocate one bit from the remaining bits to the entry that gives the largest decrease of the MSE.
2) Reduce the number of remaining bits by one.
3) If the number of remaining bits is zero, exits, and if not, return to 1.

Next, for the fixed bit allocation, we design feedback filters to minimize MSE. All we have to do is to find the filter that minimizes

$$g(F_i[z]) := \frac{\|G[z]F_i[z]\|^2}{\left(1 - \frac{c_P\|F_i[z]-1\|}{\sqrt{3}2^{b_i}}\right)^2} \tag{13}$$

for each $i = 1, 2, \ldots, n$. However, it is not that easy to analytically find the optimal $F_i[z]$. Here, we numerically evaluate $g(F_i[z])$ as follows.

For a fixed $\gamma_{\eta_i} := \|F_i[z] - 1\|$, we find the minimum of $\|G[z]F_i[z]\|$ by solving the following optimization problem:

$$\min_{R[z]\in RH_\infty, \gamma_\epsilon} \gamma_\epsilon \tag{14}$$

subject to $R[\infty] = 1$,

$$\|G[z]F_i[z]\| \leq \gamma_\epsilon \tag{15}$$

$$\|F_i[z] - 1\| \leq \gamma_{\eta_i} \tag{16}$$

where RH_∞ is the set of stable proper rational functions with real coefficients. This problem can be cast into a convex optimization problem [14], which can be solved numerically and efficiently by using a numerical solver like CVX[15].

Once the optimal filters are obtained, we find the bit allocation again. We resort to the greedy algorithm to obtain our bit allocation, although the optimality of the greedy algorithm is not guaranteed unlike the uniform quantizers. Then, for the given bit allocation, we find the optimal filters that minimize (13).

We repeat the successive bit allocation and filter design described above until a convergence criterion is satisfied. Since our algorithm monotonically decreases MSE, its convergence is assured but its limiting point is not optimal in general. It is noted that the convergence speed does not matter so much, since the bit allocation and feedback filters are supposed to be designed offline.

IV. A Design example

Due to the lack of space, we can show only one design example of a rotary inverted pendulum detailed in [16]. We remark that similar improvements of MSE have been confirmed in other examples.

In continuous-time, the state of the rotary inverted pendulum is

$$x^T(t) = [\phi(t), \theta(t), \dot{\phi}(t), \dot{\theta}(t)],$$

where $\theta(t)$ is the yaw angle of the arm and $\phi(t)$ is the pitch angle of the the pendulum. The torque $u(t)$ is applied to

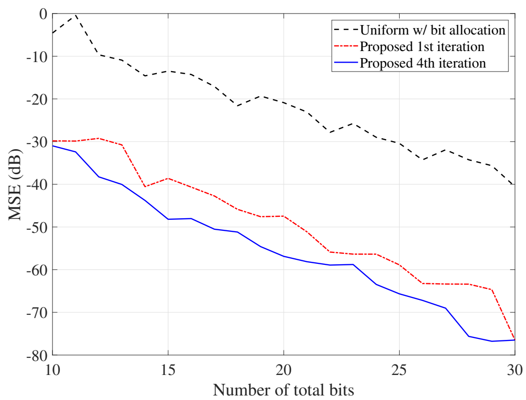

Fig. 3. Empirical MSEs for different numbers of total bits.

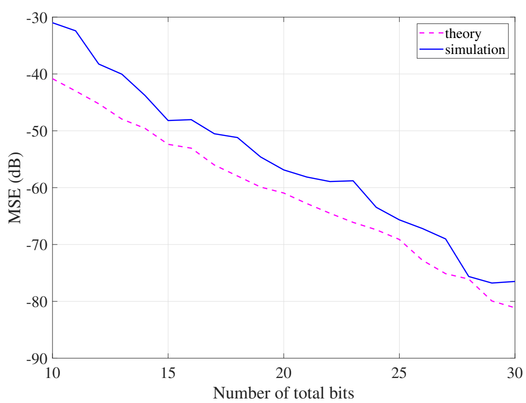

Fig. 4. Empirical and theoretical MSEs.

actuate the pendulum. If $\phi(t) = 0$, then the pendulum is balanced in the inverted position.

The linearized continuous system is discretized with the sampling period 0.01. The state feedback control with gain is set to be

$$[k_1, k_2, k_3, k_4] = [57.2598, 6.0910, 6.2562, 3.4953].$$

The angle of the pendulum and the angle of the arm is physically limited. On the other hand, the angular velocity of the pendulum and the angular velocity of the arm depend on the control law. We evaluate their maximum values by numerical simulations without quantization. To avoid overflows, doubles of them are used. Then, the saturation levels are set to be

$$[\pi/2, \pi, 3.3492, 5.4236].$$

Fig. 3 compares empirical MSEs of feedback quantizers and uniform quantizers for different numbers of total bits. For the optimal bit allocation for uniform quantizers, we design feedback quantizers and find bit allocation based on the greedy algorithm. After the first iteration, the feedback quantizers exhibit better MSE performance than the uniform quantizers. After the 4th iteration, we could not find any significant improvement in MSE.

Fig. 4 compares the empirical MSE after the 4th iteration with its corresponding theoretical MSE. Although the whiteness assumption does not hold true, the empirical MSE is similar to the theoretical MSE, which justifies our design.

V. Conclusions

We have proposed a heuristic design that iteratively optimizes bit allocation and feedback filters of feedback quantizers. A numerical example shows that our design exhibits better performance than the conventional design.

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