Design Method of Directional GenLOT with Trend Vanishing Moments

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Abstract—**This work summerizes some theoretical properties of trend vanishing moments (TVMs), which the authors have defined in a previous work. Then, the relation between the TVM condition and the polyphase order of 2-D non-separable GenLOT are clarified. Consequently, the design procedure is generalized in terms of the polyphase order from the authors' previous works. The TVM is an extention of the 1-D vanishing moment (VM) to 2-D one and can be regarded as an alternative of the directional vanishing moments (DVMs). While the conventional DVM condition requires for the moments to vanish along lines in the frequency domain and restricts the direction to a rational factor, the TVM condition imposes the moments only point-wisely and the direction can be steered flexibly. In order to verify the significance of directional GenLOTs with two-order TVM, the capability of sparse representation for trend surfaces is shown.**

I. INTRODUCTION

Most applications of signal processing highly depend on the choice of signal representations. From this background, discussions on advanced transforms would be significant for stimulating other related fields of research such as statistical modeling of signals, optimization processes, implementation issues and so forth. This paper focuses on a recent development of our proposed 2-D orthogonal transforms.

Image transforms have found a variety of applications such as image coding, denoising, feature extraction, compressive sampling and so forth [1]–[3]. Especially, orthogonal transforms are desirable for many applications since the orthogonality yields the energy conservation between the original signals and transform coefficients, and significantly reduces complexity in the mathematical handling of algorithms compared with the biorthogonal bases and frames.

In the classical approach to transform an image, two 1-D transforms are separately applied to the vertical and horizontal direction. For example, JPEG, MPEG-1/2 and H.264/AVC consist of the 2-D separable DCT, and JPEG2000 adopts separable biorthogonal discrete wavelet transforms (DWTs) [1]. Such separable transforms have disadvantage in representing diagonal geometric structure such as diagonal edges and textures, no matter whether it is orthogonal or biorthogonal. The transform coefficients are prone to scatter around several high-frequency subbands. For proper handling of such geometric structures, non-separable processing is essential.

There have been several attempts to represent, approximate and compress images by introducing non-separable transforms [4]–[7]. The curvelet is one of the successful 2-D transforms, which can efficiently approximate curve-like edges [7]. It, however, is overcomplete and initially developed in continuous domain, and constructing a fast discretized orthogonal curvelet-like transform is an open problem. In the article [5], Do and Vetterli start with discrete-domain construction of filter banks for producing an alternative directional multiresolution analysis framework. In order to compactly approximate piecewise smooth images with smooth contours, a DVM condition is introduced. The DVM condition, however, is restricted its direction to rational angles [5], [6].

As previous works, we have proposed a lattice structure of non-separable linear-phase paraunitary filter banks (LPPUFBs) [8], derived the two-order VM condition [9], [10], and introduced the block-wise implementation with a directional design approach [11]. Since the structure is based on DCT, it is viewed as a 2-D non-separable extension of GenLOT [12]. The block-wise implementation serves variability of basis images without any violation to the orthogonality. Consequently, a boundary operation for size-limitation and compatibility with the block DCT are yielded. The 2-D non-separable GenLOT shows prospective significance. The article [13] deals with the directional design issue by introducing a novel directional vanishing moments, *the trend vanishing moments (TVMs)*, then in the article [14], some theoretical properties of TVMs for general orthogonal transforms are investigated. In this work, we further attempt to clarify the relation between the polyphase order and the TVMs. As a result, we show more sophisticated design procedure from one given in [13].

II. REVIEW OF 2-D NON-SEPARABLE GENLOT

In this section, let us review the lattice structure of 2-D non-separable GenLOTs.

A. Symbols and Notations

In this paper, variable vectors in 2-D z-transform domain and discrete-spatial Fourier transform domain are denoted by $\mathbf{z} = (z_\mathrm{y}, z_\mathrm{x})^T \in \mathscr{C}^2$ and $\boldsymbol{\omega} = (\omega_\mathrm{y}, \omega_\mathrm{x})^T \in \mathscr{R}^2$, respectively. M_y and M_x are reserved for decimation factors respectively in the vertical and horizontal direction, which are assumed to be even. Let us define $\mathbf{M} = \begin{pmatrix} M_\mathrm{y} & 0 \\ 0 & M_\mathrm{x} \end{pmatrix}$. Then, the total decimation factor is given by $M = |\det(\mathbf{M})| = M_\mathrm{y} \times M_\mathrm{x}$. We reserve $\mathbf{E}_0$ to the $M \times M$ symmetric orthonormal transform given directly through the 2-D separable DCT, where each basis image is aligned into an $M \times 1$ vector and is arrayed into the following form:

$$\mathbf{E}_0^T = \begin{pmatrix} \mathbf{B}_{\mathrm{ee}} & \mathbf{B}_{\mathrm{oo}} & \mathbf{B}_{\mathrm{oe}} & \mathbf{B}_{\mathrm{eo}} \end{pmatrix},$$

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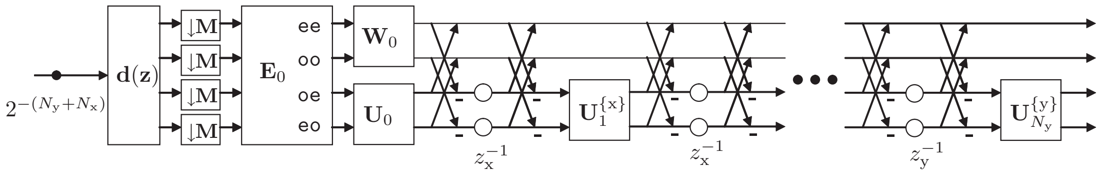


Fig. 1. Lattice structure of a 2-D non-separable GenLOT (forward transform).

where $\mathbf{B}_{ee}$, $\mathbf{B}_{oo}$, $\mathbf{B}_{oe}$ and $\mathbf{B}_{eo}$ are $M \times M/4$ matrices consisting of columnized basis images, and the superscript T denotes the matrix transposition. The first and second subscript denote the symmetry in the vertical and horizontal direction, where 'e' and 'o' describe even- and odd-symmetry, respectively.

Through this paper, notations $((x))_M$ and $\lfloor x \rfloor$ denote the modulo x of M and the largest integer less than or equal to x, respectively, and symbols $\mathbf{o}_m$, $\mathbf{O}_m$, $\mathbf{I}_m$ and $\mathbf{J}_m$ are reserved for the $m \times 1$ null vector, the $m \times m$ null, identity and counter-identity matrix, respectively, where the subscript m is omitted unless it is significant. A product of sequential matrices is denoted by $\prod_{n=1}^{N} \mathbf{A}_n = \mathbf{A}_N \mathbf{A}_{N-1} \cdots \mathbf{A}_2 \mathbf{A}_1$, where we define $\prod_{n=a}^{b} \mathbf{A}_n = \mathbf{I}$ for $b < a$ for the sake of concise representation. As well, $\sum_{n=a}^{b} \mathbf{A}_n = \mathbf{O}$ for $b < a$.

B. Lattice Structure of 2-D Non-Separable GenLOTs

In the article [8], we have shown a method to construct multidimensional LPPUFBs with a lattice structure, then Lu Gan *et.al.* have shown the reduced parameterization [15].

Figure 1 illustrates the lattice structure for 2-D signals with the parameter reduction, i.e. a 2-D non-separable GenLOT. The corresponding polyphase matrix of order $[N_y, N_x]$ is represented by the following product form:

$$\mathbf{E}(\mathbf{z}) = \prod_{n_y=1}^{N_y} \left\{ \mathbf{R}_{n_y}^{\{y\}} \mathbf{Q}(z_y) \right\} \cdot \prod_{n_x=1}^{N_x} \left\{ \mathbf{R}_{n_x}^{\{x\}} \mathbf{Q}(z_x) \right\} \cdot \mathbf{R}_0 \mathbf{E}_0, \quad (1)$$

where

$$\mathbf{Q}(z_d) = \frac{1}{2} \begin{pmatrix} \mathbf{I}_{\frac{M}{2}} & \mathbf{I}_{\frac{M}{2}} \\ \mathbf{I}_{\frac{M}{2}} & -\mathbf{I}_{\frac{M}{2}} \end{pmatrix} \begin{pmatrix} \mathbf{I}_{\frac{M}{2}} & \mathbf{O}_{\frac{M}{2}} \\ \mathbf{O}_{\frac{M}{2}} & z_d^{-1} \mathbf{I}_{\frac{M}{2}} \end{pmatrix} \begin{pmatrix} \mathbf{I}_{\frac{M}{2}} & \mathbf{I}_{\frac{M}{2}} \\ \mathbf{I}_{\frac{M}{2}} & -\mathbf{I}_{\frac{M}{2}} \end{pmatrix},$$

$$\mathbf{R}_0 = \begin{pmatrix} \mathbf{W}_0 & \mathbf{O}_{\frac{M}{2}} \\ \mathbf{O}_{\frac{M}{2}} & \mathbf{U}_0 \end{pmatrix}, \ \mathbf{R}_n^{\{d\}} = \begin{pmatrix} \mathbf{I}_{\frac{M}{2}} & \mathbf{O}_{\frac{M}{2}} \\ \mathbf{O}_{\frac{M}{2}} & \mathbf{U}_n^{\{d\}} \end{pmatrix}.$$

Matrices $\mathbf{W}_0$, $\mathbf{U}_0$ and $\mathbf{U}_{n_d}^{\{d\}}$ are orthonormal matrices of size $M/2 \times M/2$ and freely controlled. Equation (1) guarantees the orthonormality and linear-phase property. The support region of each analysis (or synthesis) filter results in $L_y \times L_x = (N_y + 1)M_y \times (N_x + 1)M_x$.

By denoting an analysis filter bank as a vector $\mathbf{H}(\mathbf{z}) = (H_0(\mathbf{z}),\ H_1(\mathbf{z}),\ \cdots,\ H_{M-1}(\mathbf{z}))^T$ and defining a 2-D delay chain of size $M \times 1$ by

$$[\mathbf{d}(\mathbf{z})]_\ell = z_y^{-((\ell))_{M_y}} \cdot z_x^{-\lfloor \ell / M_y \rfloor}, \quad (2)$$

we have the relation $\mathbf{H}(\mathbf{z}) = \mathbf{E}(\mathbf{z}^{\mathbf{M}})\mathbf{d}(\mathbf{z})$, where $[\mathbf{x}]_\ell$ means the ℓ-th element of vector $\mathbf{x}$. For diagonal $\mathbf{M}$, $\mathbf{z}^{\mathbf{M}}$ equals to $(z_y^{M_y}, z_x^{M_x})^T$.

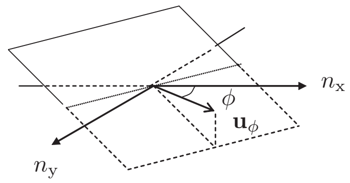


Fig. 2. A trend surface proportional to $(n_y \sin\phi + n_x \cos\phi)$.

III. Trend Vanishing Moment (TVM)

This section reviews the definition of TVM and summerizes some of the significant properties [14].

A. Definition of TVM

Let $0^0 = 1$ for the sake of concise representation, and let us refer to $H_0(\mathbf{z})$ as a scaling filter and $H_k(\mathbf{z})$ for $k \geq 1$ as a wavelet filter.

Definition 1 (Trend Vanishing Moments of Order P)**.** *We say that a filter bank has P-order TVM along the direction* $\mathbf{u}_\phi = (\sin\phi, \cos\phi)^T$ *if trend moments* $\mu_{k,\phi}^{(p)}$ *of all wavelet filters up to* $p = (P-1)$ *vanishes, i.e.*

$$0 = \mu_{k,\phi}^{(p)} = \sum_{\mathbf{n} \in \mathscr{Z}^2} h_k[\mathbf{n}] \sum_{q=0}^{p} \binom{p}{q} (n_y \sin\phi)^{p-q} (n_x \cos\phi)^q =$$
$$(-j)^p \sum_{q=0}^{p} \binom{p}{q} \sin^{p-q}\phi \cos^q\phi \frac{\partial^p}{\partial \omega_y^{p-q} \partial \omega_x^q} H_k\left(e^{j\omega^T}\right) \Bigg|_{\omega = \mathbf{o}} \quad (3)$$

for all $k = 1, 2, \cdots, M-1$ *and* $p = 0, 1, \cdots, P-1$*, where* $\mathbf{n} = [n_y, n_x]^T$ *and* $h_k[\mathbf{n}]$ *is the impulse response of the k-th analysis filter* $H_k(\mathbf{z})$*, i.e. the k-th basis image for the paraunitary case.*

See Appendix A for the significance of Eq. (3). The one-order TVM is identical to the classical one-order VM and guarantees the no-DC-leakage property. It can be verified that the wavelet filters with the two-order TVM annihilate piecewise one-order trend surfaces in the direction $\mathbf{u}_\phi$, i.e. functions proportional to $(n_y \sin\phi + n_x \cos\phi)$ as shown in Fig. 2.

Note that the classical 2-D VM constraint requires simultaneous imposition of independent vanishing moment conditions for every monomial $n_y^{(p-q)} n_x^q$ for $0 \leq q \leq p$ and $0 \leq p \leq P-1$ [16], [17]. We have experienced that such a strict constraint often disturbs the directional design of filters [9], [10].

The original directional vanishing moment (DVM) requires to impose for the moments to vanish not only at $\omega = \mathbf{o}$ but

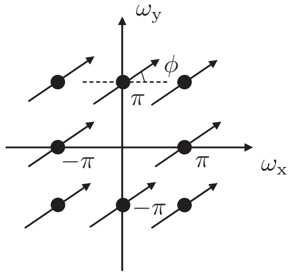


Fig. 3. Illustration of TVM condition on a scaling filter for $M_y = M_x = 2$, where the dots shows the frequency points at which the response and derivatives in the direction ϕ become null.

also along the line $\mathbf{u}_\phi^T \omega = \mathbf{o}$ [5], [6]. The DVM imposition, which puts the factor $(1-\mathbf{z}^{\mathbf{u}})^P$ with an integer vector $\mathbf{u} = c\mathbf{u}_\phi$, to every wavelet filter is restrictive in terms of the steerability. Note that the TVM can be steered flexibly.

B. TVM Condition for Scaling Filter

The TVM condition in Definition 1 is equivalently represented in terms of the scaling filter $H_0(\mathbf{z})$ as described in the following theorem.

Theorem 1. *For paraunitary filter banks, the condition in Eq.* (4) *holds if and only if Eq.* (3) *is satisfied.*

$$0 = \sum_{q=0}^{p} \binom{p}{q} \sin^{p-q}\phi \cos^q \phi \frac{\partial^p}{\partial \omega_y^{p-q} \partial \omega_x^q} H_0\left(e^{j\omega^T}\right)\Bigg|_{\omega=\omega_\ell} \tag{4}$$

for $p = 0, 1, \cdots, P-1$ *and all* $(M-1)$ *aliasing frequencies, i.e.* $\omega_\ell = 2\pi \mathbf{M}^{-T}\mathbf{k}_\ell$ *for* $\mathbf{k}_\ell \in \mathscr{N}\left(\mathbf{M}^T\right) \setminus \{\mathbf{o}\}$, *where* $\mathscr{N}(\mathbf{N}) = \left\{\mathbf{Nx} \in \mathscr{Z}^2 | \mathbf{x} \in [0,1)^2\right\}$ *and* $\mathbf{k}_0 = \mathbf{o}$ *[18].*

See [14] for the proof. Note that the diagonal decimation factor $\mathbf{M}$ yields the aliasing frequencies given by

$$\omega_\ell = \begin{pmatrix} \frac{2\pi}{M_y}((\ell))_{M_y} \\ \frac{2\pi}{M_x}\left\lfloor \frac{\ell}{M_y} \right\rfloor \end{pmatrix}, \ \ell = 1, 2, \cdots, M-1.$$

Figure 3 illustrates the TVM condition on a scaling filter in the frequency domain for $M_y = M_x = 2$.

C. Polyphase Matrix Expression of the TVM Condition

In the article [19], the VM conditions are derived for the lattice structure of 1-D M-channel LPPUFBs. The authors discuss the issue in terms of polyphase matrix representation. The following lemma shows that a similar expression can be derived for the TVM case.

Lemma 1. *For an FIR paraunitary filter bank, the TVM condition of order P in Eq.* (3) *is represented in terms of the polyphase matrix* $\mathbf{E}(\mathbf{z})$ *by*

$$c_p \mathbf{a}_M = \mathbf{m}_\phi^{(p)} = \sum_{q=0}^{p} \binom{p}{q} \sin^{p-q}\phi \cos^q \phi \frac{\partial^p}{\partial \omega_y^{p-q} \partial \omega_x^q} \mathbf{E}\left(\mathbf{z}^{\mathbf{M}}\right) \mathbf{d}(\mathbf{z}) \Bigg|_{\mathbf{z}=\mathbf{1}} \tag{5}$$

for $p = 0, 1, \cdots, P-1$, *where* c_p *is an arbitrary constant,* $\mathbf{1} = (1, 1, \cdots, 1)^T$ *and* $\mathbf{a}_m$ *is the* $m \times 1$ *vector defined by* $\mathbf{a}_m = (1, 0, \cdots, 0)^T$, *where* $\mathbf{m}_\phi^{(p)}$ *is the* $M \times 1$ *vector of which k-th element is given by* $\mu_{k,\phi}^{(p)}$.

See [14] for the proof.

IV. TVM Conditions for Lattice Parameters

In this section, we further proceed to obtain the TVM condition of order two for the lattice parameters of 2-D non-separable GenLOTs.

A. One-Order TVM Condition for Lattice Parameters

The one-order TVM is prerequisite for higher-order TVM and the condition for the lattice parameter is given as the following theorem.

Theorem 2. *For filter banks given by Eq.* (1), *the one-order TVM condition results in the following form of parameter matrix* $\mathbf{W}_0$ *with* $c_0 = \sqrt{M}$:

$$\mathbf{W}_0 = \begin{pmatrix} 1 & \mathbf{o}^T \\ \mathbf{o} & \overline{\mathbf{W}}_0 \end{pmatrix}, \tag{6}$$

where $\overline{\mathbf{W}}_0$ *is an arbitrary orthonormal matrix of size* $(M/2-1) \times (M/2-1)$.

This one-order TVM condition guarantees the no-DC-leakage property [20]. See [14] for the proof.

B. Two-Order TVM Condition for Lattice Parameters

The TVM condition in Eq. (5) of order two is rewritten by the lattice parameters as the following theorem.

Theorem 3. *For filter banks given by Eq.* (1), *the two-oder TVM condition is represented, in addition to Eq.* (6), *by the following equation:*

$$\begin{aligned}\mathbf{o} = {} & M_y \sin\phi \sum_{k_y=1}^{N_y} \prod_{n_y=k_y}^{N_y} \mathbf{U}_{n_y}^{\{y\}} \cdot \mathbf{a}_{\frac{M}{2}} \\ & + M_x \cos\phi \prod_{n_y=1}^{N_y} \mathbf{U}_{n_y}^{\{y\}} \cdot \sum_{k_x=1}^{N_x} \prod_{n_x=k_x}^{N_x} \mathbf{U}_{n_x}^{\{x\}} \cdot \mathbf{a}_{\frac{M}{2}} \\ & + \prod_{n_y=1}^{N_y} \mathbf{U}_{n_y}^{\{y\}} \cdot \prod_{n_x=1}^{N_x} \mathbf{U}_{n_x}^{\{x\}} \cdot \mathbf{U}_0 \mathbf{b}_\phi,\end{aligned} \tag{7}$$

where

$$\mathbf{b}_\phi = \mathbf{b}_y \sin\phi + \mathbf{b}_x \cos\phi, \tag{8a}$$

$$\mathbf{b}_d = \frac{2}{\sqrt{M}} \begin{pmatrix} \mathbf{B}_{oe}^T \\ \mathbf{B}_{eo}^T \end{pmatrix} \begin{pmatrix} \mathbf{I}_{\frac{M}{2}} \\ -\mathbf{J}_{\frac{M}{2}} \end{pmatrix} \mathbf{v}_d, \ d \in \{y, x\}, \tag{8b}$$

where $[\mathbf{v}_y]_\ell = \frac{M_y-1}{2} - ((\ell))_{M_y}$, $[\mathbf{v}_x]_\ell = \frac{M_x-1}{2} - \left\lfloor \frac{\ell}{M_y} \right\rfloor$ *for* $\ell = 0, 1, \cdots, M/2-1$.

Proof: For $p=1$, the following relation is derived:

$$\mathbf{m}_\phi^{(1)} = \left\{\sin\phi\frac{\partial}{\partial z_{\rm y}}+\cos\phi\frac{\partial}{\partial z_{\rm x}}\right\}\mathbf{E}(\mathbf{z}^{\mathbf{M}})\mathbf{d}(\mathbf{z})\Big|_{\mathbf{z}=\mathbf{1}}$$

$$= \frac{\sqrt{M}}{2}\left[\sin\phi\left\{M_{\rm y}\begin{pmatrix}-N_{\rm y}\mathbf{I}\\ \sum_{k_{\rm y}=1}^{N_{\rm y}}\prod_{n_{\rm y}=k_{\rm y}}^{N_{\rm y}}\mathbf{U}_{n_{\rm y}}^{\{\rm y\}}\end{pmatrix}\mathbf{a}_{\frac{M}{2}} + \begin{pmatrix}-(M_{\rm y}-1)\mathbf{a}_{\frac{M}{2}}\\ \prod_{n_{\rm y}=1}^{N_{\rm y}}\mathbf{U}_{n_{\rm y}}^{\{\rm y\}}\cdot\prod_{n_{\rm x}=1}^{N_{\rm x}}\mathbf{U}_{n_{\rm x}}^{\{\rm x\}}\cdot\mathbf{U}_0\mathbf{b}_{\rm y}\end{pmatrix}\right\}\right.$$

$$\left.+\cos\phi\left\{M_{\rm x}\begin{pmatrix}-N_{\rm x}\mathbf{I}\\ \prod_{n_{\rm y}=1}^{N_{\rm y}}\mathbf{U}_{n_{\rm y}}^{\{\rm y\}}\cdot\sum_{k_{\rm x}=1}^{N_{\rm x}}\prod_{n_{\rm x}=k_{\rm x}}^{N_{\rm x}}\mathbf{U}_{n_{\rm x}}^{\{\rm x\}}\end{pmatrix}\mathbf{a}_{\frac{M}{2}} + \begin{pmatrix}-(M_{\rm x}-1)\mathbf{a}_{\frac{M}{2}}\\ \prod_{n_{\rm y}=1}^{N_{\rm y}}\mathbf{U}_{n_{\rm y}}^{\{\rm y\}}\cdot\prod_{n_{\rm x}=1}^{N_{\rm x}}\mathbf{U}_{n_{\rm x}}^{\{\rm x\}}\cdot\mathbf{U}_0\mathbf{b}_{\rm x}\end{pmatrix}\right\}\right]. \quad (9)$$

From Lemma 1, the upper half and the lower half vector of $\mathbf{m}_\phi^{(1)}$ should be $c_1\mathbf{a}_{M/2}$ and null, respectively. The condition for the upper half one is always satisfied. Therefore, the two-order TVM condition results in the Eq (7) from the condition for the lower half vector. ∎

V. Relation between TVM and Polyphase Order

This work newly contributes to discuss the relation between the two-order TVM and the polyphase order $[N_{\rm y}, N_{\rm x}]$. Note that the length of $\mathbf{b}_\phi$ is given by

$$\|\mathbf{b}_\phi\| = \sqrt{\frac{1}{3}\left\{\left(M_{\rm y}^2-1\right)\sin^2\phi+(M_{\rm x}^2-1)\cos^2\phi\right\}}. \quad (10)$$

A. Necessary Condition with respect to the Polyphase Order

Eq. (10) allows us to derive a necessary condition for the polyphase order $[N_{\rm y}, N_{\rm x}]$.

Theorem 4. *Paraunitary filter banks of decimation factor $M=|\det\mathbf{M}|\geq 2$ in the form Eq.* (1) *require polyphase order $[N_{\rm y}, N_{\rm x}]$ such that $(N_{\rm y}+N_{\rm x})>1$ to hold the two-order TVM except for the following singular angles:*

$$\phi = \begin{cases}\tan^{-1}\left(\pm\sqrt{\frac{M_{\rm x}^2-1}{2M_{\rm y}^2+1}}\right), & N_{\rm y}=1, N_{\rm x}=0,\\ \cot^{-1}\left(\pm\sqrt{\frac{M_{\rm y}^2-1}{2M_{\rm x}^2+1}}\right), & N_{\rm y}=0, N_{\rm x}=1.\end{cases} \quad (11)$$

Proof:

1) For $N_{\rm y}=N_{\rm x}=0$: Bases are non-overlapping and the TVM condition in Eq. (7) reduces to

$$\mathbf{o}=\mathbf{U}_0\mathbf{b}_\phi.$$

Since $\mathbf{U}_0$ is orthonormal, the right hand side preserves the length of $\mathbf{b}_\phi$, which is not null. Thus, the condition never holds for any angle ϕ.

2) For $N_{\rm y}=1$ and $N_{\rm x}=0$: Equation (7) reduces to

$$\mathbf{o}=M_{\rm y}\sin\phi\,\mathbf{a}_{\frac{M}{2}}+\mathbf{U}_0\mathbf{b}_\phi$$

which can hold only if $\|\mathbf{b}_\phi\|=M_{\rm y}|\sin\phi|$. From Eq. (10), the singular angles result in Eq. (11).

3) For $N_{\rm y}=0$ and $N_{\rm x}=1$: Equation (7) reduces to

$$\mathbf{o}=M_{\rm x}\cos\phi\,\mathbf{a}_{\frac{M}{2}}+\mathbf{U}_0\mathbf{b}_\phi$$

which can hold only if $\|\mathbf{b}_\phi\|=M_{\rm x}|\cos\phi|$. From Eq. (10), the singular angles result in Eq. (11). ∎

B. Sufficient Condition with respect to the Polyphase Order

In the followings, we further discuss the relation between the two-order TVM and the polyphase order. We here assume that $M_{\rm y}=M_{\rm x}=\sqrt{M}\geq 2$, i.e. $M=M_{\rm y}M_{\rm x}\geq 4$, for the sake of simplification.

Theorem 5. *Paraunitary filter banks of diagonal decimation factors $M_{\rm y}=M_{\rm x}=\sqrt{M}\geq 2$ in the form Eq.* (1) *can hold the two-order TVM for any angle in the following specified range when the corresponding condition is satisfied:*

1) *For $\phi\in[-\pi/4,\pi/4]$, the horizontal polyphase order $N_{\rm x}$ is larger than or equal to two.*
2) *For $\phi\in[\pi/4,3\pi/4]$, the vertical polyphase order $N_{\rm y}$ is larger than or equal to two.*

See Appendices B and C for the proof.

VI. Design Procedure

In this section, let us show a design procedure of 2-D directional GenLOT (DirLOTs) with two-order TVM. The discussion here refines our preceding work [13]. Note here that the one-order TVM imposition, which is mandatory to realize the two-order TVM, is simply achieved on our DCT-based lattice structure as expressed by Eq. (6). The following discussion contributes to realize the rest of the conditions.

As a preliminary, let us introduce a utility matrix.

- $\mathbf{V}_\lambda[\mathbf{x}]$: an orthonormal parameter matrix which rotates the vector $\mathbf{x}$ to keep the angle with $\mathbf{a}_{M/2}$ to λ. This matrix is parameterized as follows:

$$\mathbf{V}_\lambda[\mathbf{x}]=\mathbf{A}_\lambda\mathbf{P}[\mathbf{x}], \quad (12)$$

where $\mathbf{P}[\mathbf{x}]$ is a planer rotation or Householder matrix which maps vector $\mathbf{x}$ to vector $\mathbf{a}_{M/2}$, and $\mathbf{A}_\lambda$ is an orthonormal parameter matrix of size $M/2\times M/2$, which can be factorized as

$$\mathbf{A}_\lambda=\begin{pmatrix}1 & \mathbf{o}^T\\ \mathbf{o} & \overline{\mathbf{A}}\end{pmatrix}\begin{pmatrix}\cos\theta_{\frac{M}{2}-1} & \mathbf{o}^T & -\sin\theta_{\frac{M}{2}-1}\\ \mathbf{o} & \mathbf{I} & \mathbf{o}\\ \sin\theta_{\frac{M}{2}-1} & \mathbf{o}^T & \cos\theta_{\frac{M}{2}-1}\end{pmatrix}\times\cdots$$

$$\times\begin{pmatrix}\cos\theta_2 & 0 & -\sin\theta_2 & \mathbf{o}^T\\ 0 & 1 & 0 & \mathbf{o}^T\\ \sin\theta_2 & 0 & \cos\theta_2 & \mathbf{o}^T\\ \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{I}\end{pmatrix}\begin{pmatrix}\cos\theta_1 & -\sin\theta_1 & \mathbf{o}^T\\ \sin\theta_1 & \cos\theta_1 & \mathbf{o}^T\\ \mathbf{o} & \mathbf{o} & \mathbf{I}\end{pmatrix}$$

with an arbitrary orthonormal matrix $\overline{\mathbf{A}}$ of size $(M/2-1)\times(M/2-1)$ such that

$$\cos\theta_1\cos\theta_2\cdots\cos\theta_{M/2-1}=\cos\lambda.$$

Especially, for $M_{\rm y}=M_{\rm x}=\sqrt{M}=2$, we have the form

$$\mathbf{A}_\lambda=\begin{pmatrix}1 & 0\\ 0 & \mu\end{pmatrix}\begin{pmatrix}\cos\lambda & \mp\sin\lambda\\ \pm\sin\lambda & \cos\lambda\end{pmatrix},$$

where $\mu \in \{-1,1\}$. It is also proven that $\mathbf{b}_\phi = -\begin{pmatrix}\sin\phi\\ \cos\phi\end{pmatrix}$, $\mathbf{b}_\phi^{\{x\}} = -\frac{1}{2}\begin{pmatrix}\tan\phi\\ 1\end{pmatrix}$ and $\mathbf{b}_\phi^{\{y\}} = -\frac{1}{2}\begin{pmatrix}1\\ \cot\phi\end{pmatrix}$.

The followings show concrete design procedures of generating DirLOTs with two-order TVM for $M_y = M_x = \sqrt{M}$.

A. For $\phi \in [-\pi/4, \pi/4]$ and $N_x \geq 2$

For the sake of consistent representation, we here temporarily define $\mathbf{U}_0^{\{x\}} = \mathbf{U}_0$.

Step 1 Give a direction ϕ and let $\bar{\mathbf{x}}_3$ as given in Tab. I.

Step 2 Impose parameter matrices $\mathbf{U}_{N_x-2}^{\{x\}}$ and $\mathbf{U}_{N_x-1}^{\{x\}}$ to be the following form:

$$\mathbf{U}_{N_x-2}^{\{x\}} = \mathbf{V}_\lambda[\bar{\mathbf{x}}_3],$$
$$\mathbf{U}_{N_x-1}^{\{x\}} = \mathbf{V}_0\left[-\mathbf{a}_{\frac{M}{2}} - \mathbf{U}_{N_x-2}^{\{x\}}\bar{\mathbf{x}}_3\right].$$

in addition to the imposition of the form in Eq. (6) on $\mathbf{W}_0$, where angle λ given by

$$\lambda = \cos^{-1}\left(1 - \frac{\|\bar{\mathbf{x}}_3\|^2}{2}\right) + \cos^{-1}\left(\frac{\|\bar{\mathbf{x}}_3\|}{2}\right). \quad (13)$$

Step 3 Optimize parameter matrices $\mathbf{W}_0$, $\mathbf{U}_0$, $\{\mathbf{U}_{n_x}^{\{x\}}\}_{n_x=1}^{N_x}$ and $\{\mathbf{U}_{n_y}^{\{y\}}\}_{n_y=1}^{N_y}$ for minimizing a given cost function under the constraint $\|\bar{\mathbf{x}}_3\| \leq 2$. When $(N_y + N_x) < 4$, this constraint is always met.

B. For $\phi \in [\pi/4, 3\pi/4]$ and $N_y \geq 2$

For the sake of consistent representation, we here temporarily define $\mathbf{U}_0^{\{y\}} = \mathbf{U}_{N_x}^{\{x\}}$.

Step 1 Give a direction ϕ and let $\bar{\mathbf{x}}_3$ as given in Tab. II.

Step 2 Impose parameter matrices $\mathbf{U}_{N_y-2}^{\{y\}}$ and $\mathbf{U}_{N_y-1}^{\{y\}}$ to be the following form:

$$\mathbf{U}_{N_y-2}^{\{y\}} = \mathbf{V}_\lambda[\bar{\mathbf{x}}_3],$$
$$\mathbf{U}_{N_y-1}^{\{y\}} = \mathbf{V}_0\left[-\mathbf{a}_{\frac{M}{2}} - \mathbf{U}_{N_y-2}^{\{y\}}\bar{\mathbf{x}}_3\right].$$

in addition to the imposition of the form in Eq. (6) on $\mathbf{W}_0$, where angle λ given by Eq. (13)

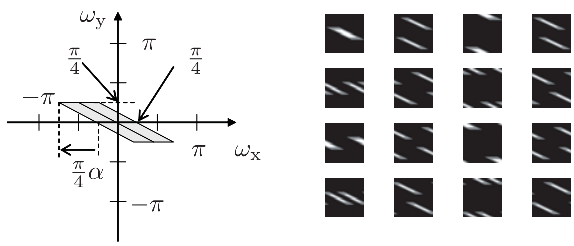
Step 3 Optimize parameter matrices $\mathbf{W}_0$, $\mathbf{U}_0$, $\{\mathbf{U}_{n_x}^{\{x\}}\}_{n_x=1}^{N_x}$ and $\{\mathbf{U}_{n_y}^{\{y\}}\}_{n_y=1}^{N_y}$ for minimizing a given cost function under the constraint $\|\bar{\mathbf{x}}_3\| \leq 2$. When $(N_y + N_x) < 4$, this constraint is always met.

VII. Design Examples

In order to verify the significance of our proposed design procedure, let us show a design example.

A. Directional Design Specification

In 2-D filter design, ideal filter characteristics are highly dependent on the frequency characteristics of a given signal array. In this work, we give frequency specifications by using symmetric parallel-piped (SPD) with a unimodular matrix, which controls the passband shape while avoiding the aliasing frequencies [18].



(a) Deformation of passband region (b) Specifications of magnitude responses

Fig. 4. An example of passband region deformation for ideal lowpass filters, and an example set of magnitude response specification for $M_y = M_x = 4$, $d = x$ and $\alpha = -2.0$, where the white, black and gray regions represent the pass, stop and transition bands, respectively.

Let us define a frequency magnitude response of an ideal lowpass filter $H_{I0}(\mathbf{z})$ as

$$\left|H_{I0}\left(e^{j\boldsymbol{\omega}^T}\right)\right| = \begin{cases}\sqrt{M}, & \boldsymbol{\omega} \in \mathrm{SPD}\left\{\pi(\mathbf{M}\mathbf{A}_d(\alpha))^{-T}\right\}\\ 0, & \boldsymbol{\omega} \in [-\pi,\pi)^2 \backslash \mathrm{SPD}\left\{\pi(\mathbf{M}\mathbf{A}_d(\alpha))^{-T}\right\}\end{cases}$$

where $\mathrm{SPD}\{\mathbf{V}\} = \left\{\mathbf{V}\mathbf{x} | \mathbf{x} \in [-1,1)^2\right\}$, $\mathbf{M}$ is the decimation matrix and $\mathbf{A}_d(\alpha)$ is a unimodular matrix defined by

$$\mathbf{A}_d(\alpha) = \begin{cases}\begin{pmatrix}1 & 0\\ -\alpha & 1\end{pmatrix}, & d = y\\ \begin{pmatrix}1 & -\alpha\\ 0 & 1\end{pmatrix}, & d = x\end{cases}.$$

α is a parameter which controls the deformation of passband region as shown in Fig. 4 (a).

For the other filters, we first split the passband region of the lowpass filter into positive half and negative half region in the deforming direction as shown in Fig. 4 (a), and then modulate each half region with aliasing frequencies, i.e. integer multiple points of π/M_y and π/M_x while maintaining the frequency support symmetry of passband regions. Figure 4 (b) shows an example set of magnitude response specification for $M_y = M_x = 4$, $d = x$ and $\alpha = -2.0$, where the white, black and gray regions represent the pass, stop and transition bands, respectively. Each square represents the frequency support in $[-\pi,\pi)^2$ and the center corresponds to DC.

B. Design Examples with Two-Order TVM

In order to verify the significance of the proposal, let us show a design example for $N_y = N_x = 2$. In the article [11], we only consider the one-order VM condition for designing directional GenLOTs. In this section, we show a design example with two-order TVM optimized for the same specification and cost function as those used in the article [11].

We here use the accumulated error energy

$$\sum_{k=0}^{M-1}\left\{\int_{\boldsymbol{\omega}\in\Omega_p\cap\Omega_s}\left|H_{Ik}(e^{j\boldsymbol{\omega}^T}) - H_k(e^{j\boldsymbol{\omega}^T})\right|^2 d\boldsymbol{\omega}\right\}$$

as the cost function [20] and then used the genetic algorithm function 'ga' in MATLAB R2010a for the optimization process, where Ω_p and Ω_s denote specified regions of passband

TABLE I

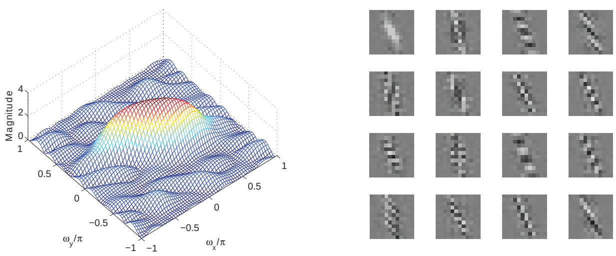
$\bar{\mathbf{x}}_3$ FOR $\phi \in [-\pi/4, \pi/4]$, WHERE $\mathbf{a}$ MEANS $\mathbf{a}_{M/2}$, AND $\overline{\mathbf{U}}$ DENOTES $\mathbf{U}_2^{\{x\}}\mathbf{U}_1^{\{x\}}\mathbf{U}_0$ FOR $N_x = 2$ AND $\mathbf{U}_{N_x}^{\{x\}}\mathbf{U}_{N_x-1}^{\{x\}}\mathbf{U}_{N_x-2}^{\{x\}}$ FOR $N_x > 2$.

N_y	N_x		
	2	3	≥ 4
0	$\mathbf{b}_\phi^{\{x\}}$	$\mathbf{a}+\mathbf{U}_0\mathbf{b}_\phi^{\{x\}}$	$\left(\mathbf{I}+\sum_{k_x=1}^{N_x-3}\prod_{n_x=k_x}^{N_x-3}\mathbf{U}_{n_x}^{\{x\}}\right)\mathbf{a}+\prod_{n_x=1}^{N_x-3}\mathbf{U}_{n_x}^{\{x\}}\cdot\mathbf{U}_0\mathbf{b}_\phi^{\{x\}}$
1	$\tan\phi\overline{\mathbf{U}}^T\mathbf{a}+\mathbf{b}_\phi^{\{x\}}$	$\tan\phi\overline{\mathbf{U}}^T\mathbf{a}+\mathbf{a}+\mathbf{U}_0\mathbf{b}_\phi^{\{x\}}$	$\tan\phi\overline{\mathbf{U}}^T\mathbf{a}+\left(\mathbf{I}+\sum_{k_x=1}^{N_x-3}\prod_{n_x=k_x}^{N_x-3}\mathbf{U}_{n_x}^{\{x\}}\right)\mathbf{a}+\prod_{n_x=1}^{N_x-3}\mathbf{U}_{n_x}^{\{x\}}\cdot\mathbf{U}_0\mathbf{b}_\phi^{\{x\}}$
≥ 2	$\tan\phi\overline{\mathbf{U}}^T\mathbf{v}^{\{y\}}+\mathbf{b}_\phi^{\{x\}}$	$\tan\phi\overline{\mathbf{U}}^T\mathbf{v}^{\{y\}}+\mathbf{a}+\mathbf{U}_0\mathbf{b}_\phi^{\{x\}}$	$\tan\phi\overline{\mathbf{U}}^T\mathbf{v}^{\{y\}}+\left(\mathbf{I}+\sum_{k_x=1}^{N_x-3}\prod_{n_x=k_x}^{N_x-3}\mathbf{U}_{n_x}^{\{x\}}\right)\mathbf{a}+\prod_{n_x=1}^{N_x-3}\mathbf{U}_{n_x}^{\{x\}}\cdot\mathbf{U}_0\mathbf{b}_\phi^{\{x\}}$

TABLE II

$\bar{\mathbf{x}}_3$ FOR $\phi \in [\pi/4, 3\pi/4]$, WHERE $\mathbf{a}$ MEANS $\mathbf{a}_{M/2}$.

N_x	N_y		
	2	3	≥ 4
0	$\mathbf{b}_\phi^{\{y\}}$	$\mathbf{a}+\mathbf{U}_0\mathbf{b}_\phi^{\{y\}}$	$\left(\mathbf{I}+\sum_{k_y=1}^{N_y-3}\prod_{n_y=k_y}^{N_y-3}\mathbf{U}_{n_y}^{\{y\}}\right)\mathbf{a}+\prod_{n_y=1}^{N_y-3}\mathbf{U}_{n_y}^{\{y\}}\cdot\mathbf{U}_0\mathbf{b}_\phi^{\{y\}}$
1	$\cot\phi\mathbf{a}+\mathbf{U}_0\mathbf{b}_\phi^{\{y\}}$	$\cot\phi\mathbf{U}_1^{\{x\}}\mathbf{a}+\mathbf{a}+\mathbf{U}_1^{\{x\}}\mathbf{U}_0\mathbf{b}_\phi^{\{y\}}$	$\cot\phi\prod_{n_y=1}^{N_y-3}\mathbf{U}_{n_y}^{\{y\}}\cdot\mathbf{U}_1^{\{x\}}\mathbf{a}+\left(\mathbf{I}+\sum_{k_y=1}^{N_y-3}\prod_{n_y=k_y}^{N_y-3}\mathbf{U}_{n_y}^{\{y\}}\right)\mathbf{a}$ $+\prod_{n_y=1}^{N_y-3}\mathbf{U}_{n_y}^{\{y\}}\cdot\mathbf{U}_1^{\{x\}}\mathbf{U}_0\mathbf{b}_\phi^{\{y\}}$
≥ 2	$\cot\phi\mathbf{v}^{\{x\}}+\prod_{n_x=1}^{N_x-1}\mathbf{U}_{n_x}^{\{x\}}\cdot\mathbf{U}_0\mathbf{b}_\phi^{\{y\}}$	$\cot\phi\mathbf{U}_{N_x}^{\{x\}}\mathbf{v}^{\{x\}}+\mathbf{a}+\prod_{n_x=1}^{N_x}\mathbf{U}_{n_x}^{\{x\}}\cdot\mathbf{U}_0\mathbf{b}_\phi^{\{y\}}$	$\cot\phi\prod_{n_y=1}^{N_y-3}\mathbf{U}_{n_y}^{\{y\}}\cdot\mathbf{U}_{N_x}^{\{x\}}\mathbf{v}^{\{x\}}+\left(\mathbf{I}+\sum_{k_y=1}^{N_y-3}\prod_{n_y=k_y}^{N_y-3}\mathbf{U}_{n_y}^{\{y\}}\right)\mathbf{a}$ $+\prod_{n_y=1}^{N_y-3}\mathbf{U}_{n_y}^{\{y\}}\cdot\prod_{n_x=1}^{N_x}\mathbf{U}_{n_x}^{\{x\}}\cdot\mathbf{U}_0\mathbf{b}_\phi^{\{y\}}$



(a) $\left|H_0\left(e^{j\omega^T}\right)\right|$ (b) Basis images

Fig. 5. A design example with two-order TVMs of $\phi = \cot^{-1}\alpha$ optimized for the specification given in Fig. 4 (b), where $N_y = N_x = 2$, i.e. basis images of size 12×12.

Fig. 6. A ramp picture of size 128×128, where $\phi_x = 30.00°$. Note that the vertical index increases from top to bottom.

and stopband, respectively. Figure 5 shows a design example with two-order TVM of $\phi = \cot^{-1}\alpha = \cot^{-1}(-2.0) \sim -26.57°$, where the magnitude response of the resulting lowpass filter $H_0(\mathbf{z})$ and basis images are shown. It can be observed from Fig. 5 that the TVM condition properly coexists with the directional design.

VIII. Experimental Results

In this section, let us verify the capability of sparse representation of DirLOTs with two-order TVM. As a measure to show the performance of sparse representation, we define a sparsity ratio R_x by

$$R_x = \frac{\sum_{k=0}^{M-1}\|y_k[\mathbf{m}]\|_0}{\|x[\mathbf{n}]\|_0}, \tag{14}$$

where M is the number of subbands, $x[\mathbf{n}]$ and $y_k[\mathbf{m}]$ denote the original picture and the k-th subband coefficient array, respectively. The symbol $\|\cdot\|_0$ means the 0-norm of the argument, i.e. the number of nonzero elements. A transform with smaller R_x is better in terms of sparse representation w.r.t. a given picture $x[\mathbf{n}]$.

Note that, in the following evaluation, we artificially generate gray scale pictures of which intensity is double precision in the range $[0, 1]$, adopt double precision computation for all operations, and numerically define null elements by value x such that $|x| \leq 10^{-15}$.

A. Simulation for Ramp Picture Rotation

In this simulation, we compare the sparsity ratios among the DirLOT shown in Fig. 5, 4×4 2-D DCT and 2-D 2-level 9/7 transform for ramp pictures of size 128×128 as shown in Fig. 6. Figure 7 shows the variation of the sparsity ratios for ramp pictures with direction ϕ_x ranging from $-45.0°$ to $135.0°$. The TVM direction ϕ of DirLOT is fixed to $\phi = \cot^{-1}\alpha = \cot^{-1}(-2.0) \sim -26.57°$.

From Fig. 7, it is observed that the TVM realizes sparse representation of the ramp picture of which direction is identical to that of the TVM. Note here that the reason why the sparsity ratio R_x is not null for the direction $\phi = \phi_x$ is mainly due to the lowpass (scaling) component and the boundary effect.

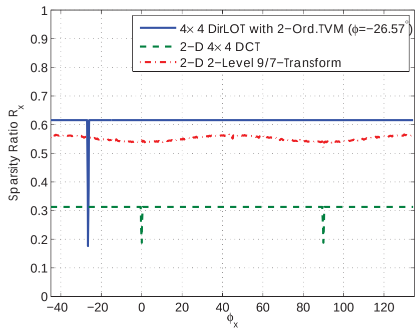


Fig. 7. Characteristics of sparsity ratio R_x against for directions of trend surface in ramp pictures, where the direction of TVM is fixed to $\phi_x = 26.57°$.

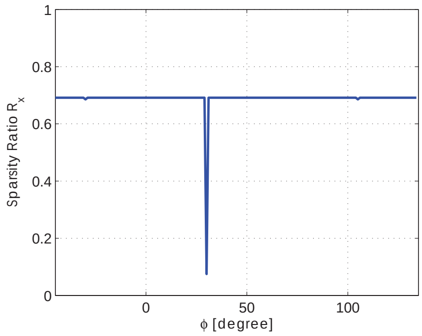


Fig. 8. Characteristics of sparsity ratio R_x against for directions of two-order TVMs, where the direction of input picture is fixed to $\phi_x = 30.00°$ and the 3-level DWT structure is adopted.

B. Simulation for TVM Rotation

In this simulation, we compare the sparsity ratios among different TVM directions. We adopt the 3-level hierarchical DWT structure of 2×2-decomposition GenLOTs. In order to remove the effect of the directional specification, the shape control parameter α is fixed to 0 for every design, i.e. a square region of support is specified for the passband of every filter. The polyphase order $[N_y, N_x]$ is selected as $[0,2]$ for $\phi \in [-\pi/4, \pi/4)$ and $[2,0]$ for $\phi \in [\pi/4, 3\pi/4)$, respectively. The ramp picture, i.e. trend surface picture, of size 128×128 shown in Fig. 6 is used, where the ramp direction is fixed to $\phi_x = 30.00°$. From Fig. 8, it is observed that there is a spiky drop at $\phi = 30.00°$, which is identical to the direction of the given trend surface.

IX. Conclusions

In this work, we introduced the trend vanishing moments for 2-D non-separable transforms and summarized some of the theoretical properties. Then, we investigated the relation between the polyphase order of non-separable GenLOTs and two-order TVM. Consequently, we proposed a novel design procedure for non-separable GenLOTs with TVM, which is generalized in terms of the polyphase order from our previous works. The significance of the two-order TVM was verified by showing the capability of sparse representation for several ramp pictures.

As future works, we will investigate the adaptive control of local basis, the fast hardware-friendly implementation and the applications to image coding, processing and analysis.

Appendix A
Significance of the TVM Condition in Eq. (3)

The condition in Eq. (3) for wavelet filters is equivalent to the VM condition for the horizontal direction in the rotated frequency plane $\hat{\omega} = \mathbf{S}_\phi^{-1}\omega$, i.e.

$$\frac{\partial^p}{\partial \hat{\omega}_x^p} H_\phi\left(e^{j\hat{\omega}^T}\right)\Big|_{\hat{\omega}=\mathbf{o}} = \frac{\partial^p}{\partial \hat{\omega}_x^p} H\left(e^{j(\mathbf{S}_\phi \hat{\omega})^T}\right)\Big|_{\hat{\omega}=\mathbf{o}} = 0 \quad \text{(A.1)}$$

where $\mathbf{S}_\phi$ is the 2×2 rotation matrix with angle ϕ defined by $\mathbf{S}_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$ and $\hat{\omega} = (\hat{\omega}_y, \hat{\omega}_x)^T$.

The p-th derivative along with the horizontal direction in the rotated frequency plane is represented by

$$\frac{\partial^p}{\partial \hat{\omega}_x^p} H_\phi\left(e^{j\hat{\omega}^T}\right) = \frac{\partial^p}{\partial \hat{\omega}_x^p} H\left(e^{j(\mathbf{S}_\phi \hat{\omega})^T}\right)$$
$$= \sum_{q=0}^{p} \binom{p}{q} \sin^{p-q}\phi \cos^q\phi \frac{\partial^p}{\partial \omega_y^{p-q} \partial \omega_x^q} H\left(e^{j\omega^T}\right). \quad \text{(A.2)}$$

Appendix B
Proof of Case 1 in Theorem 5

Proof: For direction $\phi \in [-\pi/4, \pi/4]$, $\cos\phi \neq 0$. Thus, the condition in Eq. (7) can be rewritten as

$$\mathbf{o} = \tan\phi \sum_{k_y=1}^{N_y} \prod_{n_y=k_y}^{N_y} \mathbf{U}_{n_y}^{\{y\}} \cdot \mathbf{a}_{\frac{M}{2}} + \prod_{n_y=1}^{N_y} \mathbf{U}_{n_y}^{\{y\}} \cdot \sum_{k_x=1}^{N_x} \prod_{n_x=k_x}^{N_x} \mathbf{U}_{n_x}^{\{x\}} \cdot \mathbf{a}_{\frac{M}{2}}$$
$$+ \prod_{n_y=1}^{N_y} \mathbf{U}_{n_y}^{\{y\}} \cdot \prod_{n_x=1}^{N_x} \mathbf{U}_{n_x}^{\{x\}} \cdot \mathbf{U}_0 \mathbf{b}_\phi^{\{x\}}, \quad \text{(B.1)}$$

where $\mathbf{b}_\phi^{\{x\}} = \mathbf{b}_\phi / \sqrt{M}\cos\phi$, of which length is given by

$$\left\|\mathbf{b}_\phi^{\{x\}}\right\| = \frac{\|\mathbf{b}_\phi\|}{\sqrt{M}\cos\phi} = \sqrt{\frac{M-1}{3M}(\tan^2\phi + 1)}. \quad \text{(B.2)}$$

Note that the length is bounded for $M \geq 4$ as $1/2 \leq \left\|\mathbf{b}_\phi^{\{x\}}\right\| < \sqrt{2/3}$ from the facts that $0 \leq \tan^2\phi \leq 1$ for $\phi \in [-\pi/4, \pi/4]$ and $(M-1)/M < 1$.

a) For $N_y = 0$ and $N_x = 2$: Equation (B.1) reduces to

$$\mathbf{o} = \mathbf{a}_{\frac{M}{2}} + \mathbf{U}_1^{\{x\}} \mathbf{a}_{\frac{M}{2}} + \mathbf{U}_1^{\{x\}} \mathbf{U}_0 \mathbf{b}_\phi^{\{x\}}$$

which is solvable when $\|\mathbf{b}_\phi^{\{x\}}\| \leq 2$ from the bilateral triangle rule [19]. Since $\|\mathbf{b}_\phi^{\{x\}}\| < \sqrt{2/3}$, we can find a solution.

b) For $N_y = 0$ and $N_x = 3$: Equation (B.1) reduces to

$$\mathbf{o} = \mathbf{a}_{\frac{M}{2}} + \mathbf{U}_2^{\{x\}} \mathbf{a}_{\frac{M}{2}} + \mathbf{U}_2^{\{x\}} \mathbf{U}_1^{\{x\}} \left(\mathbf{a}_{\frac{M}{2}} + \mathbf{U}_0 \mathbf{b}_\phi^{\{x\}}\right)$$

which is solvable from the bilateral triangle rule when $\|\mathbf{a}_{M/2} + \mathbf{U}_0 \mathbf{b}_\phi^{\{x\}}\| \leq 2$. Since $\max_{\mathbf{U}_0} \|\mathbf{a}_{M/2} + \mathbf{U}_0 \mathbf{b}_\phi^{\{x\}}\| < 1 + \sqrt{2/3} < 2$, we can find a solution.

c) For $N_y = 0$ and $N_x \geq 4$: Equation (B.1) reduces to

$$\mathbf{o} = \mathbf{a}_{\frac{M}{2}} + \mathbf{U}^{\{x\}}_{N_x-1}\mathbf{a}_{\frac{M}{2}} + \mathbf{U}^{\{x\}}_{N_x-1}\mathbf{U}^{\{x\}}_{N_x-2}\times \left(\mathbf{a}_{\frac{M}{2}} + \sum_{k_x=1}^{N_x-3}\prod_{n_x=k_x}^{N_x-3}\mathbf{U}^{\{x\}}_{n_x}\cdot\mathbf{a}_{\frac{M}{2}} + \prod_{n_x=1}^{N_x-3}\mathbf{U}^{\{x\}}_{n_x}\cdot\mathbf{U}_0\mathbf{b}^{\{x\}}_{\phi}\right)$$

which is solvable from the bilateral triangle rule under the constraint that the length of the last term in the right hand side is less than or equal to two, i.e.

$$\left\|\mathbf{a}_{\frac{M}{2}} + \sum_{k_x=1}^{N_x-3}\prod_{n_x=k_x}^{N_x-3}\mathbf{U}^{\{x\}}_{n_x}\cdot\mathbf{a}_{\frac{M}{2}} + \prod_{n_x=1}^{N_x-3}\mathbf{U}^{\{x\}}_{n_x}\cdot\mathbf{U}_0\mathbf{b}^{\{x\}}_{\phi}\right\| \leq 2.$$

By induction with the two previous cases, the left hand side can be proven to vanish with appropriate control of the parameter matrices. Therefore, the above constraint is valid.

d) For $N_y = 1$ and $N_x = 2$: Equation (B.1) reduces to

$$\mathbf{o} = \tan\phi\,\mathbf{a}_{\frac{M}{2}} + \mathbf{U}^{\{x\}}_{2}\left(\mathbf{a}_{\frac{M}{2}} + \mathbf{U}^{\{x\}}_{1}\mathbf{a}_{\frac{M}{2}} + \mathbf{U}^{\{x\}}_{1}\mathbf{U}_0\mathbf{b}^{\{x\}}_{\phi}\right)$$

or equivalently

$$\mathbf{o} = \mathbf{a}_{\frac{M}{2}} + \mathbf{U}^{\{x\}}_{1}\mathbf{a}_{\frac{M}{2}} + \mathbf{U}^{\{x\}}_{1}\mathbf{U}_0\left(\tan\phi\,\overline{\mathbf{U}}^{T}\mathbf{a}_{\frac{M}{2}} + \mathbf{b}^{\{x\}}_{\phi}\right),$$

where $\overline{\mathbf{U}} = \mathbf{U}^{\{x\}}_{2}\mathbf{U}^{\{x\}}_{1}\mathbf{U}_0$. Note that $\overline{\mathbf{U}}$ is orthonormal and freely chosen because all factors are orthonormal and $\mathbf{U}^{\{x\}}_{2}$ is free from the other parameter matrices. The above condition can hold when

$$\left\|\tan\phi\,\mathbf{a}_{\frac{M}{2}} + \overline{\mathbf{U}}\mathbf{b}^{\{x\}}_{\phi}\right\| \leq 2.$$

The maximum of the left hand side is bounded as

$$\max_{\overline{\mathbf{U}}}\left\|\tan\phi\,\mathbf{a}_{\frac{M}{2}} + \overline{\mathbf{U}}\mathbf{b}^{\{x\}}_{\phi}\right\| = |\tan\phi| + \left\|\mathbf{b}^{\{x\}}_{\phi}\right\|.$$

Since $|\tan\phi| \leq 1$ and $\|\mathbf{b}^{\{x\}}_{\phi}\| < \sqrt{2/3}$, this case is solvable.

e) For $N_y = 1$ and $N_x = 3$: Equation (B.1) reduces to

$$\mathbf{o} = \tan\phi\,\mathbf{a}_{\frac{M}{2}} + \mathbf{U}^{\{x\}}_{3}\left(\mathbf{a}_{\frac{M}{2}} + \sum_{k_x=1}^{2}\prod_{n_x=k_x}^{2}\mathbf{U}^{\{x\}}_{n_x}\cdot\mathbf{a}_{\frac{M}{2}} + \prod_{n_x=1}^{2}\mathbf{U}^{\{x\}}_{n_x}\cdot\mathbf{U}_0\mathbf{b}^{\{x\}}_{\phi}\right)$$

or equivalently

$$\mathbf{o} = \mathbf{a}_{\frac{M}{2}} + \mathbf{U}^{\{x\}}_{2}\mathbf{a}_{\frac{M}{2}} + \mathbf{U}^{\{x\}}_{2}\mathbf{U}^{\{x\}}_{1}\left(\tan\phi\,\overline{\mathbf{U}}^{T}\mathbf{a}_{\frac{M}{2}} + \mathbf{a}_{\frac{M}{2}} + \mathbf{U}_0\mathbf{b}^{\{x\}}_{\phi}\right),$$

where $\overline{\mathbf{U}} = \mathbf{U}^{\{x\}}_{3}\mathbf{U}^{\{x\}}_{2}\mathbf{U}^{\{x\}}_{1}$. $\overline{\mathbf{U}}$ is an arbitrary orthonormal matrix and is freely chosen because $\mathbf{U}^{\{x\}}_{3}$ is free from the other parameter matrices. The above condition can hold from the bilateral triangle rule under the constraint

$$\left\|\tan\phi\,\mathbf{a}_{\frac{M}{2}} + \overline{\mathbf{U}}\mathbf{a}_{\frac{M}{2}} + \overline{\mathbf{U}}\mathbf{U}_0\mathbf{b}^{\{x\}}_{\phi}\right\| \leq 2.$$

Note that the left hand side can vanish for $\phi \neq 0$ since we can find a set of parameter matrices such that $\mathbf{o} = \sqrt{M}\sin\phi\,\mathbf{a}_{M/2} + \sqrt{M}\cos\phi\,\overline{\mathbf{U}}\mathbf{a}_{M/2} + \overline{\mathbf{U}}\mathbf{U}_0\mathbf{b}_{\phi}$ except for $\phi = 0$. For $\phi = 0$, the maximum of the left hand side is bounded as

$$\max_{\mathbf{U}_0}\left\|\mathbf{a}_{\frac{M}{2}} + \mathbf{U}_0\mathbf{b}^{\{x\}}_{0}\right\| = 1 + \left\|\mathbf{b}^{\{x\}}_{0}\right\| < 2,$$

since $\|\mathbf{b}^{\{x\}}_{\phi}\| < \sqrt{2/3}$. Thus, the constraint is valid.

f) For $N_y = 1$ and $N_x \geq 4$: Equation (B.1) reduces to

$$\mathbf{o} = \tan\phi\,\mathbf{a}_{\frac{M}{2}} + \sum_{k_x=1}^{N_x}\prod_{n_x=k_x}^{N_x}\mathbf{U}^{\{x\}}_{n_x}\cdot\mathbf{a}_{\frac{M}{2}} + \prod_{n_x=1}^{N_x}\mathbf{U}^{\{x\}}_{n_x}\cdot\mathbf{U}_0\mathbf{b}^{\{x\}}_{\phi}$$

or equivalently

$$\mathbf{o} = \mathbf{a}_{\frac{M}{2}} + \mathbf{U}^{\{x\}}_{N_x-1}\mathbf{a}_{\frac{M}{2}} + \mathbf{U}^{\{x\}}_{N_x-1}\mathbf{U}^{\{x\}}_{N_x-2}\Big(\tan\phi\,\overline{\mathbf{U}}^{T}\mathbf{a}_{\frac{M}{2}} + \mathbf{a}_{\frac{M}{2}} + \sum_{k_x=1}^{N_x-3}\prod_{n_x=k_x}^{N_x-3}\mathbf{U}^{\{x\}}_{n_x}\cdot\mathbf{a}_{\frac{M}{2}} + \prod_{n_x=1}^{N_x-3}\mathbf{U}^{\{x\}}_{n_x}\cdot\mathbf{U}_0\mathbf{b}^{\{x\}}_{\phi}\Big),$$

where $\overline{\mathbf{U}} = \mathbf{U}^{\{x\}}_{N_x}\mathbf{U}^{\{x\}}_{N_x-1}\mathbf{U}^{\{x\}}_{N_x-2}$, which is orthonormal and freely controllable. The above condition is solvable from the bilateral triangle rule under the constraint

$$\left\|\tan\phi\,\mathbf{a}_{\frac{M}{2}} + \overline{\mathbf{U}}\left(\mathbf{a}_{\frac{M}{2}} + \sum_{k_x=1}^{N_x-3}\prod_{n_x=k_x}^{N_x-3}\mathbf{U}^{\{x\}}_{n_x}\cdot\mathbf{a}_{\frac{M}{2}} + \prod_{n_x=1}^{N_x-3}\mathbf{U}^{\{x\}}_{n_x}\cdot\mathbf{U}_0\mathbf{b}^{\{x\}}_{\phi}\right)\right\| \leq 2.$$

By induction with the previous two cases, the left hand side can be proven to vanish by properly controlling the parameter matrices. Thus, the above constraint is valid.

g) For $N_y \geq 2$ and $N_x = 2$: Equation (B.1) reduces to

$$\mathbf{o} = \tan\phi\,\mathbf{v}^{\{y\}} + \mathbf{U}^{\{x\}}_{2}\left(\mathbf{a}_{\frac{M}{2}} + \mathbf{U}^{\{x\}}_{1}\mathbf{a}_{\frac{M}{2}} + \mathbf{U}^{\{x\}}_{1}\mathbf{U}_0\mathbf{b}^{\{x\}}_{\phi}\right),$$

or equivalently

$$\mathbf{o} = \mathbf{a}_{\frac{M}{2}} + \mathbf{U}^{\{x\}}_{1}\mathbf{a}_{\frac{M}{2}} + \mathbf{U}^{\{x\}}_{1}\mathbf{U}_0\left(\tan\phi\,\overline{\mathbf{U}}^{T}\mathbf{v}^{\{y\}} + \mathbf{b}^{\{x\}}_{\phi}\right),$$

where $\overline{\mathbf{U}} = \mathbf{U}^{\{x\}}_{2}\mathbf{U}^{\{x\}}_{1}\mathbf{U}_0$, which is orthonormal and freely controllable, and

$$\mathbf{v}^{\{y\}} = \mathbf{a}_{\frac{M}{2}} + \sum_{k_y=2}^{N_y}\left(\prod_{n_y=1}^{k_y-1}\mathbf{U}^{\{y\}}_{n_y}\right)^{T}\mathbf{a}_{\frac{M}{2}}. \qquad \text{(B.3)}$$

The above condition can hold from the bilateral triangle rule under the constraint

$$\left\|\tan\phi\,\mathbf{v}^{\{y\}} + \overline{\mathbf{U}}\mathbf{b}^{\{x\}}_{\phi}\right\| \leq 2.$$

This constraint is valid for any $\phi \in [-\pi/4, \pi/4]$ because the vector $\mathbf{v}^{\{y\}}$ can be null with appropriate control of the parameter matrices $\{\mathbf{U}^{\{y\}}_{n_y}\}$, e.g. by canceling the component vectors or constituting a polygon. Thus, the minimum of the left hand side is less than $\|\mathbf{b}^{\{x\}}_{\phi}\| < \sqrt{2/3} < 2$. Consequently, this case is solvable.

h) For $N_y \geq 2$ and $N_x = 3$: Equation (B.1) is rewritten as

$$\mathbf{o} = \tan\phi\,\mathbf{v}^{\{y\}} + \mathbf{U}^{\{x\}}_{3}\left(\mathbf{a}_{\frac{M}{2}} + \mathbf{U}^{\{x\}}_{2}\mathbf{a}_{\frac{M}{2}} + \mathbf{U}^{\{x\}}_{2}\mathbf{U}^{\{x\}}_{1}\mathbf{a}_{\frac{M}{2}} + \mathbf{U}^{\{x\}}_{2}\mathbf{U}^{\{x\}}_{1}\mathbf{U}_0\mathbf{b}^{\{x\}}_{\phi}\right)$$

or equivalently

$$\mathbf{o} = \mathbf{a}_{\frac{M}{2}} + \mathbf{U}^{\{x\}}_{2}\mathbf{a}_{\frac{M}{2}} + \mathbf{U}^{\{x\}}_{2}\mathbf{U}^{\{x\}}_{1}\left(\tan\phi\,\overline{\mathbf{U}}^{T}\mathbf{v}^{\{y\}} + \mathbf{a}_{\frac{M}{2}} + \mathbf{U}_0\mathbf{b}^{\{x\}}_{\phi}\right),$$

where $\overline{\mathbf{U}} = \mathbf{U}_3^{\{x\}}\mathbf{U}_2^{\{x\}}\mathbf{U}_1^{\{x\}}$, which is orthonormal and freely controllable. The above condition can hold from the bilateral triangle rule under the constraint

$$\left\|\tan\phi\mathbf{v}^{\{y\}} + \overline{\mathbf{U}}\left(\mathbf{a}_{\frac{M}{2}} + \mathbf{U}_0\mathbf{b}_\phi^{\{x\}}\right)\right\| \leq 2.$$

This constraint is valid since $\mathbf{v}^{\{y\}}$ can vanish and the remaining terms constitute a vector of length less than $\max_{\mathbf{U}_0}\|\mathbf{a}_{M/2} + \mathbf{U}_0\mathbf{b}_\phi^{\{x\}}\| < 1 + \sqrt{2/3} < 2$. Therefore, this case is solvable.

i) For $N_y \geq 2$ and $N_x \geq 4$: Equation (B.1) is rewritten as

$$\mathbf{o} = \tan\phi\mathbf{v}^{\{y\}} + \sum_{k_x=1}^{N_x}\prod_{n_x=k_x}^{N_x}\mathbf{U}_{n_x}^{\{x\}}\cdot\mathbf{a}_{\frac{M}{2}} + \prod_{n_x=1}^{N_x}\mathbf{U}_{n_x}^{\{x\}}\cdot\mathbf{U}_0\mathbf{b}_\phi^{\{x\}},$$

or equivalently

$$\mathbf{o} = \mathbf{a}_{\frac{M}{2}} + \mathbf{U}_{N_x-1}^{\{x\}}\mathbf{a}_{\frac{M}{2}} + \mathbf{U}_{N_x-1}^{\{x\}}\mathbf{U}_{N_x-2}^{\{x\}}\left(\tan\phi\overline{\mathbf{U}}^T\mathbf{v}^{\{y\}} + \mathbf{a}_{\frac{M}{2}} + \sum_{k_x=1}^{N_x-3}\prod_{n_x=k_x}^{N_x-3}\mathbf{U}_{n_x}^{\{x\}}\cdot\mathbf{a}_{\frac{M}{2}} + \prod_{n_x=1}^{N_x-3}\mathbf{U}_{n_x}^{\{x\}}\cdot\mathbf{U}_0\mathbf{b}_\phi^{\{x\}}\right),$$

where $\overline{\mathbf{U}} = \mathbf{U}_{N_x}^{\{x\}}\mathbf{U}_{N_x-1}^{\{x\}}\mathbf{U}_{N_x-2}^{\{x\}}$, which is orthonormal and freely controllable. The above condition is solvable from the bilateral triangle rule under the constraint

$$\left\|\tan\phi\mathbf{v}^{\{y\}} + \overline{\mathbf{U}}\left(\mathbf{a}_{\frac{M}{2}} + \sum_{k_x=1}^{N_x-3}\prod_{n_x=k_x}^{N_x-3}\mathbf{U}_{n_x}^{\{x\}}\cdot\mathbf{a}_{\frac{M}{2}} + \prod_{n_x=1}^{N_x-3}\mathbf{U}_{n_x}^{\{x\}}\cdot\mathbf{U}_0\mathbf{b}_\phi^{\{x\}}\right)\right\| \leq 2.$$

Vector $\mathbf{v}^{\{y\}}$ in the left hand side can vanish and the remaining terms can constitute a closed loop by properly controlling the parameter matrices. See the proof of *Case c)*. As a result, the validity of this constraint is confirmed. ∎

APPENDIX C
PROOF OF CASE 2 IN THEOREM 5

Proof: For $\phi \in [\pi/4, 3\pi/4]$, $\sin\phi \neq 0$. Thus, the condition in Eq. (7) can be rewritten for $M_y = M_x = \sqrt{M}$ as

$$\mathbf{o} = \sum_{k_y=1}^{N_y}\prod_{n_y=k_y}^{N_y}\mathbf{U}_{n_y}^{\{y\}}\cdot\mathbf{a}_{\frac{M}{2}} + \cot\phi\prod_{n_y=1}^{N_y}\mathbf{U}_{n_y}^{\{y\}}\cdot\sum_{k_x=1}^{N_x}\prod_{n_x=k_x}^{N_x}\mathbf{U}_{n_x}^{\{x\}}\cdot\mathbf{a}_{\frac{M}{2}} + \prod_{n_y=1}^{N_y}\mathbf{U}_{n_y}^{\{y\}}\cdot\prod_{n_x=1}^{N_x}\mathbf{U}_{n_x}^{\{x\}}\cdot\mathbf{U}_0\mathbf{b}_\phi^{\{y\}}, \quad \text{(C.1)}$$

where $\mathbf{b}_\phi^{\{y\}} = \mathbf{b}_\phi/\sqrt{M}\sin\phi$ of which length is given by

$$\left\|\mathbf{b}_\phi^{\{y\}}\right\| = \frac{\|\mathbf{b}_\phi\|}{\sqrt{M}\sin\phi} = \sqrt{\frac{M-1}{3M}(1+\cot^2\phi)}. \quad \text{(C.2)}$$

The length is bounded for $M \geq 4$ as $1/2 \leq \left\|\mathbf{b}_\phi^{\{y\}}\right\| < \sqrt{2/3}$ from the facts that $0 \leq \cot^2\phi \leq 1$ for $\phi \in [\pi/4, 3\pi/4]$ and $(M-1)/M < 1$.

a) For $N_y = 2$ and $N_x = 0$: The same discussion with the proof of *Case a)* in Appendix B holds by reducing Eq. (C.1) and swapping the symbols 'x' and 'y.'

b) For $N_y = 3$ and $N_x = 0$: The same discussion with the proof of *Case b)* in Appendix B holds by reducing Eq. (C.1) and swapping the symbols 'x' and 'y.'

c) For $N_y \geq 4$ and $N_x = 0$: The same discussion with the proof of *Case c)* in Appendix B holds by reducing Eq. (C.1) and swapping the symbols 'x' and 'y.'

d) For $N_y = 2$ and $N_x = 1$: Equation (C.1) reduces to

$$\mathbf{o} = \mathbf{a}_{\frac{M}{2}} + \mathbf{U}_1^{\{y\}}\mathbf{a}_{\frac{M}{2}} + \mathbf{U}_1^{\{y\}}\mathbf{U}_1^{\{x\}}\left(\cot\phi\mathbf{a}_{\frac{M}{2}} + \mathbf{U}_0\mathbf{b}_\phi^{\{y\}}\right),$$

which can hold from the bilateral triangle rule if

$$\left\|\cot\phi\mathbf{a}_{\frac{M}{2}} + \mathbf{U}_0\mathbf{b}_\phi^{\{y\}}\right\| \leq 2.$$

It is proven that the above inequality always holds since $\|\mathbf{b}_\phi^{\{y\}}\| < \sqrt{2/3}$ and $|\cot\phi| \leq 1$. Thus, this case is solvable.

e) For $N_y = 3$ and $N_x = 1$: Equation (C.1) reduces to

$$\mathbf{o} = \mathbf{a}_{\frac{M}{2}} + \mathbf{U}_2^{\{y\}}\mathbf{a}_{\frac{M}{2}} + \mathbf{U}_2^{\{y\}}\mathbf{U}_1^{\{y\}}\times\left(\cot\phi\mathbf{U}_1^{\{x\}}\mathbf{a}_{\frac{M}{2}} + \mathbf{a}_{\frac{M}{2}} + \mathbf{U}_1^{\{x\}}\mathbf{U}_0\mathbf{b}_\phi^{\{y\}}\right), \quad \text{(C.3)}$$

which is solvable from the bilateral triangle rule under the constraint

$$\left\|\cot\phi\mathbf{U}_1^{\{x\}}\mathbf{a}_{\frac{M}{2}} + \mathbf{a}_{\frac{M}{2}} + \mathbf{U}_1^{\{x\}}\mathbf{U}_0\mathbf{b}_\phi^{\{y\}}\right\| \leq 2.$$

The left hand side can be proven to vanish for $\phi \neq \pi/2$ except for $\phi = \pi/2$. For $\phi = \pi/2$, the maximum of the left hand side is bounded as

$$\max_{\mathbf{U}_1^{\{x\}},\mathbf{U}_0}\left\|\mathbf{a}_{\frac{M}{2}} + \mathbf{U}_1^{\{x\}}\mathbf{U}_0\mathbf{b}_{\frac{\pi}{2}}^{\{y\}}\right\| = 1 + \left\|\mathbf{b}_{\frac{\pi}{2}}^{\{y\}}\right\| < 2,$$

since $\|\mathbf{b}_\phi^{\{y\}}\| < \sqrt{2/3}$. Thus, the constraint is valid.

f) For $N_y \geq 4$ and $N_x = 1$: Equation (C.1) reduces to

$$\mathbf{o} = \mathbf{a}_{\frac{M}{2}} + \mathbf{U}_{N_y-1}^{\{y\}}\mathbf{a}_{\frac{M}{2}} + \mathbf{U}_{N_y-1}^{\{y\}}\mathbf{U}_{N_y-2}^{\{y\}}\left(\cot\phi\prod_{n_y=1}^{N_y-3}\mathbf{U}_{n_y}^{\{y\}}\cdot\mathbf{U}_1^{\{x\}}\mathbf{a}_{\frac{M}{2}} + \mathbf{a}_{\frac{M}{2}} + \sum_{k_y=1}^{N_y-3}\prod_{n_y=k_y}^{N_y-3}\mathbf{U}_{n_y}^{\{y\}}\cdot\mathbf{a}_{\frac{M}{2}} + \prod_{n_y=1}^{N_y-3}\mathbf{U}_{n_y}^{\{y\}}\cdot\mathbf{U}_1^{\{x\}}\mathbf{U}_0\mathbf{b}_\phi^{\{y\}}\right),$$

which is solvable from the bilateral triangle rule under the constraint

$$\left\|\cot\phi\prod_{n_y=1}^{N_y-3}\mathbf{U}_{n_y}^{\{y\}}\cdot\mathbf{U}_1^{\{x\}}\mathbf{a}_{\frac{M}{2}} + \mathbf{a}_{\frac{M}{2}} + \sum_{k_y=1}^{N_y-3}\prod_{n_y=k_y}^{N_y-3}\mathbf{U}_{n_y}^{\{y\}}\cdot\mathbf{a}_{\frac{M}{2}} + \prod_{n_y=1}^{N_y-3}\mathbf{U}_{n_y}^{\{y\}}\cdot\mathbf{U}_1^{\{x\}}\mathbf{U}_0\mathbf{b}_\phi^{\{y\}}\right\| \leq 2.$$

By induction with the previous two cases, the left hand side can be proven to vanish by properly controlling the parameter matrices. Thus, the above constraint is valid.

g) For $N_\mathrm{y}=2$ and $N_\mathrm{x}\geq 2$: Equation (C.1) reduces to

$$\mathbf{o}=\mathbf{a}_{\frac{M}{2}}+\mathbf{U}_1^{\{\mathrm{y}\}}\mathbf{a}_{\frac{M}{2}}+\mathbf{U}_1^{\{\mathrm{y}\}}\mathbf{U}_{N_\mathrm{x}}^{\{\mathrm{x}\}}\left(\cot\phi\mathbf{v}^{\{\mathrm{x}\}}+\prod_{n_\mathrm{x}=1}^{N_\mathrm{x}-1}\mathbf{U}_{n_\mathrm{x}}^{\{\mathrm{x}\}}\cdot\mathbf{U}_0\mathbf{b}_\phi^{\{\mathrm{y}\}}\right),$$

where

$$\mathbf{v}^{\{\mathrm{x}\}}=\mathbf{a}_{\frac{M}{2}}+\sum_{k_\mathrm{x}=1}^{N_\mathrm{x}-1}\prod_{n_\mathrm{x}=k_\mathrm{x}}^{N_\mathrm{x}-1}\mathbf{U}_{n_\mathrm{x}}^{\{\mathrm{x}\}}\cdot\mathbf{a}_{\frac{M}{2}}.$$

The above condition can hold from the bilateral triangle rule under the constraint

$$\left\|\cot\phi\mathbf{v}^{\{\mathrm{x}\}}+\prod_{n_\mathrm{x}=1}^{N_\mathrm{x}-1}\mathbf{U}_{n_\mathrm{x}}^{\{\mathrm{x}\}}\cdot\mathbf{U}_0\mathbf{b}_\phi^{\{\mathrm{y}\}}\right\|\leq 2.$$

This constraint is valid for any $\phi\in[\pi/4,3\pi/4]$ because vector $\mathbf{v}^{\{\mathrm{x}\}}$ can be null by constituting a closed loop with appropriate control of parameter matrices $\{\mathbf{U}_{n_\mathrm{x}}^{\{\mathrm{x}\}}\}$. Consequently, the minimum of the left hand side shuld be less than $\|\mathbf{b}_\phi^{\{\mathrm{y}\}}\|<\sqrt{2/3}<2$.

h) For $N_\mathrm{y}=3$ and $N_\mathrm{x}\geq 2$: Equation (C.1) reduces to

$$\mathbf{o}=\mathbf{a}_{\frac{M}{2}}+\mathbf{U}_2^{\{\mathrm{y}\}}\mathbf{a}_{\frac{M}{2}}+\mathbf{U}_2^{\{\mathrm{y}\}}\mathbf{U}_1^{\{\mathrm{y}\}}\times\left(\cot\phi\mathbf{U}_{N_\mathrm{x}}^{\{\mathrm{x}\}}\mathbf{v}^{\{\mathrm{x}\}}+\mathbf{a}_{\frac{M}{2}}+\prod_{n_\mathrm{x}=1}^{N_\mathrm{x}}\mathbf{U}_{n_\mathrm{x}}^{\{\mathrm{x}\}}\cdot\mathbf{U}_0\mathbf{b}_\phi^{\{\mathrm{y}\}}\right),$$

which is solvable from the bilateral triangle rule under the constraint

$$\left\|\cot\phi\mathbf{U}_{N_\mathrm{x}}^{\{\mathrm{x}\}}\mathbf{v}^{\{\mathrm{x}\}}+\mathbf{a}_{\frac{M}{2}}+\prod_{n_\mathrm{x}=1}^{N_\mathrm{x}}\mathbf{U}_{n_\mathrm{x}}^{\{\mathrm{x}\}}\cdot\mathbf{U}_0\mathbf{b}_\phi^{\{\mathrm{y}\}}\right\|\leq 2.$$

This constraint is valid since vector $\mathbf{v}^{\{\mathrm{x}\}}$ in the left hand side can vanish and the remaining terms can constitute together a vector of length less than $\max_{\mathbf{U}_0}\|\mathbf{a}_{M/2}+\prod_{n_\mathrm{x}=1}^{N_\mathrm{x}}\mathbf{U}_{n_\mathrm{x}}^{\{\mathrm{x}\}}\cdot\mathbf{U}_0\mathbf{b}_\phi^{\{\mathrm{x}\}}\|<1+\sqrt{2/3}<2$.

i) For $N_\mathrm{y}\geq 4$ and $N_\mathrm{x}\geq 2$: Equation (C.1) is rewritten as

$$\mathbf{o}=\mathbf{a}_{\frac{M}{2}}+\mathbf{U}_{N_\mathrm{y}-1}^{\{\mathrm{y}\}}\mathbf{a}_{\frac{M}{2}}+\mathbf{U}_{N_\mathrm{y}-1}^{\{\mathrm{y}\}}\mathbf{U}_{N_\mathrm{y}-2}^{\{\mathrm{y}\}}\times\left(\cot\phi\prod_{n_\mathrm{y}=1}^{N_\mathrm{y}-3}\mathbf{U}_{n_\mathrm{y}}^{\{\mathrm{y}\}}\cdot\mathbf{U}_{N_\mathrm{x}}^{\{\mathrm{x}\}}\mathbf{v}^{\{\mathrm{x}\}}+\mathbf{a}_{\frac{M}{2}}+\sum_{k_\mathrm{y}=1}^{N_\mathrm{y}-3}\prod_{n_\mathrm{y}=k_\mathrm{y}}^{N_\mathrm{y}-3}\mathbf{U}_{n_\mathrm{y}}^{\{\mathrm{y}\}}\cdot\mathbf{a}_{\frac{M}{2}}+\prod_{n_\mathrm{y}=1}^{N_\mathrm{y}-3}\mathbf{U}_{n_\mathrm{y}}^{\{\mathrm{y}\}}\cdot\prod_{n_\mathrm{x}=1}^{N_\mathrm{x}}\mathbf{U}_{n_\mathrm{x}}^{\{\mathrm{x}\}}\cdot\mathbf{U}_0\mathbf{b}_\phi^{\{\mathrm{y}\}}\right),$$

which can hold from the bilateral triangle rule under the constraint

$$\left\|\cot\phi\prod_{n_\mathrm{y}=1}^{N_\mathrm{y}-3}\mathbf{U}_{n_\mathrm{y}}^{\{\mathrm{y}\}}\cdot\mathbf{U}_{N_\mathrm{x}}^{\{\mathrm{x}\}}\mathbf{v}^{\{\mathrm{x}\}}+\mathbf{a}_{\frac{M}{2}}+\sum_{k_\mathrm{y}=1}^{N_\mathrm{y}-3}\prod_{n_\mathrm{y}=k_\mathrm{y}}^{N_\mathrm{y}-3}\mathbf{U}_{n_\mathrm{y}}^{\{\mathrm{y}\}}\cdot\mathbf{a}_{\frac{M}{2}}+\prod_{n_\mathrm{y}=1}^{N_\mathrm{y}-3}\mathbf{U}_{n_\mathrm{y}}^{\{\mathrm{y}\}}\cdot\prod_{n_\mathrm{x}=1}^{N_\mathrm{x}}\mathbf{U}_{n_\mathrm{x}}^{\{\mathrm{x}\}}\cdot\mathbf{U}_0\mathbf{b}_\phi^{\{\mathrm{y}\}}\right\|\leq 2.$$

By induction with the previous two cases, the left hand side can be proven to vanish by properly controlling the parameter matrices. Thus, the above constraint is valid. ■

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