Consistent Sampling and Reconstruction of Signals in Noisy Under-Determined Case

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***Abstract*—We propose a sampling theorem that reconstructs a consistent signal from noisy under-determined samples. Consistency in the context of sampling theory means that the reconstructed signal yields the same measurements as the original ones. The conventional consistent sampling theorems for the under-determined case reconstruct a signal from noiseless samples in a complementary subspace L in the reconstruction space of the intersection between the reconstruction space and the orthogonal complement of the sampling space. To obtain a good reconstruction result, one has to determine L effectively. To this end, we first extend the sampling theorem for a noisy case. Since the reconstructed signal is scattered in L, we propose to reconstruct a signal so that the variance is minimized provided that the average of the signals agrees with the noiseless reconstruction. Note that the minimum variance depends on the subspace L. Therefore, we next propose to determine L so that the minimum variance is further minimized in terms of L. We show that such L can be chosen if and only if L includes a subspace determined by the noise covariance matrix. By computer simulations, we demonstrate that there is a considerable difference between the minimum and non-minimum variance reconstructions.**

I. Introduction

The best known result in sampling theory is the perfect reconstruction theorem for bandlimited signals of lowpass type [1], [2]. If we formally apply the theorem to a real signal, which is not exactly bandlimited, an orthogonal projection of the signal onto the subspace of all bandlimited signals is reconstructed. This is the best approximation to the original signal within the subspace. This observation brought sampling theory the approximation viewpoint.

Along this context, criteria to evaluate the degree of approximation play the central role. The relevant one is the minimum squared error between the reconstructed and the original signals [3]∼ [6]. This criterion can be satisfied if and only if the subspace spanned by the reconstruction functions is a subset of that spanned by the sampling ones. If we are free to design the reconstruction or sampling functions, then we can suffice the condition.

If sampling and reconstruction subspaces are fixed in advance, the relation does not hold in general. Then, we need some relaxed criterion. One of them is consistency, which requests that the reconstructed signal yields the same measurements as the original ones [7]∼[12]. This criterion was first employed in [7] for a critical sampling case. The authors clarified that the consistent reconstruction is an oblique projection onto the reconstruction space along the orthogonal complement of the sampling space. Further, they showed that the error of the reconstruction to the target function is bounded by that of the orthogonal projection to the target one. This discussion was extended to the over-sampling case in [9].

For the under-sampling case, the consistent reconstruction is not unique because of the non-zero intersection between the reconstruction space and the orthogonal complement of the sampling space. To make it unique, we need to determine a complementary subspace L in the reconstruction space of the intersection. This problem was first addressed in [10], in which the subspace L was determined using principal components given in an *a priori* manner. On the other hand, Dvorkind *et al.* proposed a method based on the minimax principle [11].

In this paper, we propose another method to determine the subspace L. To this end, we first extend the sampling theorem for noisy case, in which the reconstructed signal is distributed within the subspace L. Hence, we propose a criterion such that the variance is minimized provided that the average of the signals agrees with the noiseless consistent reconstruction. We derive a sampling theorem that reconstructs the optimal signal in the sense of the criterion. This is a natural extension of the sampling theorem in [6] since the proposed one reduces to the previous one in a special case, while the noise suppression mechanism of the proposed approach is clearly understood more than that in [6], because of the relevant expression of the optimum solution.

Based on this result, we propose a new method to determine the subspace L. The important observation is that the minimum value of the variance depends on the subspace L. Hence, we determine L so that the minimum value is further minimized in terms of L. It is clarified that such L is chosen if and only if L includes a subspace defined by the noise covariance matrix. This implies that such L is not unique in general. Under a certain condition, however, L is uniquely determined as the orthogonal complement in the reconstruction space of the intersection. By computer simulations, we show that there is a considerable difference between the minimum and non-minimum variance reconstructions.

This paper is organized as follows. In Section II, after formulation of the sampling and reconstruction problem, we quickly review the consistent sampling theorem for noiseless case. Section III extends the consistent sampling theorem for a noisy case, and propose the new sampling theorem. Section IV discusses the problem of determining the subspace L so that it minimizes the minimized variance. Section V concludes the paper.

II. Noiseless Consistent Sampling

The original input signal f is defined over a continuous domain $\mathcal{D}$ and is assumed to belong to a Hilbert space $H = H(\mathcal{D})$. The measurements of f, denoted by d_n ($n = 0, \ldots, N-1$), are given by the inner product in H of f with the sampling functions $\{\psi_n\}_{n=0}^{N-1}$, as

$$d_n = \langle f, \psi_n \rangle.$$

The N-dimensional vector consisting of d_n is denoted by $\boldsymbol{d}$. Let A_s be the operator that maps f into $\boldsymbol{d}$:

$$A_s f = \boldsymbol{d}. \tag{1}$$

The reconstructed signal $\tilde{f} \in H$ is given by a linear combination of reconstruction functions, $\{\varphi_k\}_{k=0}^{K-1}$:

$$\tilde{f} = \sum_{k=0}^{K-1} c_k \varphi_k. \tag{2}$$

The K-dimensional vector of signal coefficients c_k is denoted by $\boldsymbol{c}$. The (adjoint) reconstruction operator is defined by

$$\tilde{f} = A_r^* \boldsymbol{c}. \tag{3}$$

Let X be the $K \times N$ matrix that maps $\boldsymbol{d}$ to $\boldsymbol{c}$:

$$X\boldsymbol{d} = \boldsymbol{c}. \tag{4}$$

Then, Eqs. (1), (3), and (4) yield

$$\tilde{f} = A_r^* X A_s f. \tag{5}$$

With this formulation, the sampling problem becomes equivalent to finding a suitable matrix X so that $\tilde{f}$ satisfies some optimality criterion.

Let V_s and V_r be subspaces in H spanned by $\{\psi_n\}_{n=0}^{N-1}$ and $\{\varphi_k\}_{k=0}^{K-1}$, respectively. They are called the *sampling* and *reconstruction spaces*, respectively.

Consider the case that the sampling and reconstruction functions are determined *a priori*, and therefore V_s and V_r are also fixed. This includes the case where a piece of music recorded in a digital way is reproduced using a commercial audio player. If $V_r \not\subset V_s$, we cannot obtain the best approximation, which is the orthogonal projection of f onto V_r. A relaxation of criterion for this case is consistency, which requests that the measurements of the reconstructed signal agree with those of the original signal, and is expressed as

$$A_s \tilde{f} = A_s f. \tag{6}$$

To obtain X which satisfies Eq. (6) is the problem in the consistency sampling theorem for noiseless case.

To discuss the problem, we assume that

$$V_r + V_s^{\perp} = H, \tag{7}$$

where $V_s^{\perp}$ is the orthogonal complement of V_s. Eq. (7) means that Eq. (6) can be achieved for any f in H.

We further assume that

$$V_r \cap V_s^{\perp} \neq \{0\}, \tag{8}$$

which implies under-sampling scenario [10]. This condition means that some signals in the reconstruction space V_r are mapped to zero vector by sampling, and therefore the reconstructed signal satisfying the consistency condition is not unique. If Eq. (8) is not true, that is if $V_r \cap V_s^{\perp} = \{0\}$, then $\tilde{f}$ that satisfies Eq. (6) is unique, and given by the oblique projection of f onto V_r along $V_s^{\perp}$. Sampling theorems that reconstruct such signals are proposed for the critical sampling scenario [7] and for the over-sampling scenario [9]. As shown in [10], however, it is easy to find a case in which Eq. (8) is true. This is the reason why we assume Eq. (8).

To enforce the uniqueness on the consistent reconstruction under the condition (8), we consider a complementary subspace L in V_r of $V_r \cap V_s^{\perp}$:

$$L \dot{+} (V_r \cap V_s^{\perp}) = V_r, \tag{9}$$

where $\dot{+}$ denotes the direct sum, not necessarily orthogonal. Further, it holds that

$$L \cap V_s^{\perp} = \{0\}. \tag{10}$$

As shown in [10], consistent reconstruction is unique in L, and is characterized geometrically as follows. Eqs. (7) and (9) implies that

$$L \dot{+} V_s^{\perp} = H. \tag{11}$$

Hence, we can define an oblique projection onto L along $V_s^{\perp}$. By using this projection operator P, the consistent reconstruction in L is given by $\tilde{f} = Pf$ [10], which is obtained by the following sampling theorem:

Proposition 1: [10] Let L be a fixed complementary subspace of $V_r \cap V_s^{\perp}$ in V_r. The unique consistent reconstruction in L is obtained by Eq. (5) if and only if X is given by

$$X = (A_r^*)^{\dagger} P A_s^{\dagger} + Y - A_r A_r^{\dagger} Y A_s A_s^{\dagger}, \tag{12}$$

where $(\cdot)^{\dagger}$ denotes the Moore-Penrose generalized inverse and Y is an arbitrary linear operator (i.e., rectangular matrix) from $\mathbf{C}^N$ to $\mathbf{C}^K$.

Note that we have the degrees of freedom for the choice of the operator X. By using this, we can minimize the influence of noise in samples.

III. Noisy Consistent Sampling

Let us consider the case in which the samples are corrupted by additive noise ε_n, as

$$y_n = d_n + \varepsilon_n = \langle f, \psi_n \rangle + \varepsilon_n.$$

Its vector expression yields

$$\boldsymbol{y} = A_s f + \boldsymbol{\varepsilon},$$

where $\boldsymbol{y}$ and $\boldsymbol{\varepsilon}$ are N-dimensional vectors whose nth elements are y_n and ε_n, respectively. The additive noise ε_n is assumed to have a zero mean and the covariance matrix Q with the form

$$Q = \underset{\boldsymbol{\varepsilon}}{E}(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^T) = \sigma^2 Q_0,$$

where Q_0 is a known positive semidefinite matrix and σ is an unknown positive real number. Then, the reconstructed signal $\tilde{f}$ is given by

$$\tilde{f} = A_r^* X \boldsymbol{y} = A_r^* X A_s f + A_r^* X \boldsymbol{\varepsilon}.$$

To obtain the consistent reconstruction in an arbitrary fixed subspace L as precisely as possible, the variance should be minimized provided that the average agrees with the noiseless consistent reconstruction Pf. Therefore, let us reconstruct a signal $\tilde{f}$ such that

$$J[X] = \underset{\boldsymbol{\varepsilon}}{E} \|\tilde{f} - Pf\|^2 \tag{13}$$

is minimized under the condition that $E_{\boldsymbol{\varepsilon}} \tilde{f} = Pf$ for any f in H. This is equivalent to

$$A_r^* X A_s = P. \tag{14}$$

To analytically derive a matrix X that provides the optimal $\tilde{f}$ in the above sense, we use the three matrices defined as

$$\Psi = (I - A_s A_s^\dagger) Q (I - A_s A_s^\dagger),$$
$$\Phi = A_s A_s^\dagger + \Psi \Psi^\dagger, \quad \text{and} \quad P_t = I - Q\Psi^\dagger.$$

We denote the range and null space of a bounded operator T by $\mathcal{R}(T)$ and $\mathcal{N}(T)$, respectively. The matrix Φ is the orthogonal projection onto $\mathcal{R}(A_s) + \mathcal{R}(Q)$, to which the vector $\boldsymbol{y}$ belongs, while $Q\Psi^\dagger$ in P_t is the oblique projection whose range and nullspace are $Q\mathcal{R}(A_s)^\perp$ and $\mathcal{N}(QP_{\mathcal{R}(A_s)^\perp})$, respectively. Then, our first main result follows:

Theorem 1: The matrix X which minimizes Eq. (13) subject to Eq. (14) is given as

$$X = (A_r^*)^\dagger P A_s^\dagger P_t + Y - A_r A_r^\dagger Y \Phi. \tag{15}$$

In this case, the minimum value J_0 of $J[X]$ is given by

$$J_0 = \sigma^2 \langle P A_s^\dagger P_t Q_0, P A_s^\dagger P_t \rangle, \tag{16}$$

where $\langle \cdot, \cdot \rangle$ is the Schmidt inner product of operators [13].

Proofs of all the theorems in this paper are abbreviated because of the limited space. By comparing the predominant term in Eq. (15) to that in Eq. (12), we know that the difference is only P_t, which is applied to the latter from the right. In noiseless case, a measurement vector $\boldsymbol{d}$ stays in the subspace $\mathcal{R}(A_s)$. In noisy case, however, $\boldsymbol{y}$ sticks out from $\mathcal{R}(A_s)$ in general. This means that we can suppress noise by pushing $\boldsymbol{y}$ back into $\mathcal{R}(A_s)$. This is implemented by the oblique projection P_t. Further, it is implied that $Q\mathcal{R}(A_s)^\perp$ is the optimum direction for the noise suppression in the sense that $J[X]$ is minimized under the condition (14). Since $\boldsymbol{y}$ mostly stays in $\mathcal{R}(\Phi) = \mathcal{R}(A_s) + \mathcal{R}(Q)$, and we can show that $\mathcal{R}(P_t\Phi) = \mathcal{R}(A_s)$, $P_t\boldsymbol{y}$ belongs to $\mathcal{R}(A_s)$ in spite that $\mathcal{R}(P_t) = \mathcal{N}(QP_{\mathcal{R}(A_s)^\perp})$. The noise suppressed vector $P_t\boldsymbol{y}$ is transformed by $(A_r^*)^\dagger P A_s^\dagger$. Then, the vector $\boldsymbol{c}$ is obtained and Eq. (2) is computed.

Let us compare these results to those obtained in [6], which discussed a problem of designing the reconstruction space V_r provided the sampling space V_s. The optimum approximation of the original signal, which is the orthogonal projection, can be obtained if and only if $V_r \subset V_s$ is true. Therefore, the maximal subspace V_s was adopted for V_r, and a sampling theorem with optimum noise suppression was derived. However, we cannot always design V_r as stated in Introduction, or V_s and V_r are predetermined. One of the relaxed criteria for such cases is consistency. The reconstructed signal for this case is the oblique projection onto L along $V_s^\perp$. In the case that we can make $L \subset V_s$ hold, the oblique projection reduces to the orthogonal projection. In this sense, the results obtained above are extensions of those in [6].

It is also interesting to compare the expressions of the results in the present paper and [6]. In the latter, the oblique projection for the noise suppression was not shown explicitly. It was stated implicitly as Corollary 1. The meaning of the operator used in the expression is not always clear. On the other hand, P_t derived in this paper has the clear meaning of the oblique projection of the noise suppression. Correspondence between the noisy expression to the noiseless one is also clear. Further, the matrix P_t also plays the key role in the derivation of the minimum value of $J[X]$.

IV. Minimum Variance Consistent Sampling

In Section III, L was arbitrarily fixed. As a result, it was shown that the minimum variance of $\tilde{f}$ due to noise is given by J_0 in Eq. (16). This depends on L. It is natural to determine L so that $J_0 = J_0[L]$ is minimized for reconstructed signals to be stable. In this section, we clarify how such an L is characterized

Toward this end, we prepare a few subspaces. Let us denote $V_r \cap V_s^\perp$ by V_0. The subspaces V_r and V_s are decomposed into orthogonal direct sum as

$$V_r = V_0 \oplus V_1, \quad V_s^\perp = V_0 \oplus V_2.$$

Then, we have the direct sum decomposition of H as $H = V_0 \oplus (V_1 \dot{+} V_2)$. Let P_0 and P_1 be the oblique projection operators onto V_0 and V_1 along $V_1 \dot{+} V_2$ and $V_0 \oplus V_2$, respectively:

$$P_0 = P_{V_0, V_1 \dot{+} V_2}, \quad P_1 = P_{V_1, V_0 \oplus V_2}.$$

Theorem 2: The minimum variance J_0 is minimized in terms of L if and only if L contains $L_n = \mathcal{R}(P_1 A_s^\dagger P_t Q_0)$:

$$L \supset L_n = \mathcal{R}(P_1 A_s^\dagger P_t Q_0). \tag{17}$$

In this case, the minimum value is given as:

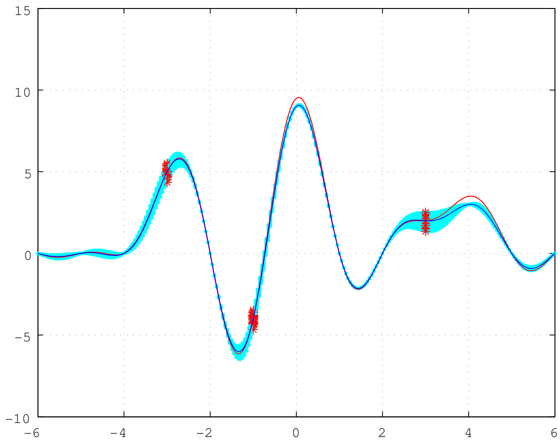
$$\min_L J_0[L] = \sigma^2 \langle P_1 A_s^\dagger P_t Q_0, P_1 A_s^\dagger P_t \rangle. \tag{18}$$

The condition (17) implies that the choice of L is not unique in general. In the following case, however, it becomes unique:

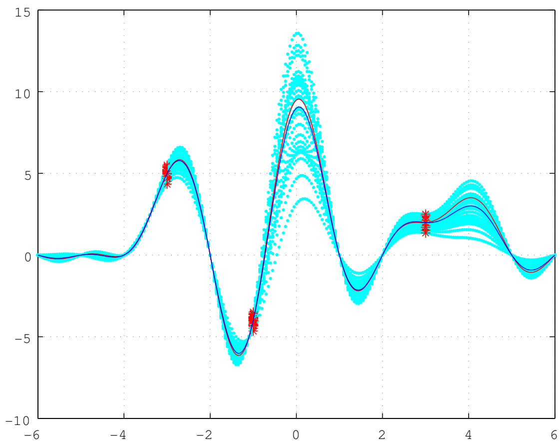
Theorem 3: The choice of L for the minimum consistent reconstruction is unique if and only if it holds that

$$\mathcal{R}(P_t Q) = \mathcal{R}(A_s). \tag{19}$$

In this case, L is given by V_1.



(a) Minimum variance reconstruction



(b) Non-minimum variance reconstruction

Fig. 1. Comparison of minimum/non-minimum variance reconstuctions.

This result agrees with that derived in [11]. That is, the subspace V_1 is optimal in the sense of both the minimum variance and the minimax criteria.

Let H be all one-variate functions bandlimited in $(-\Omega, \Omega)$ with the inner product defined as $\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$.

The reconstruction functions are given by

$$\varphi_k(x) = \frac{\Omega}{\pi}\mathrm{sinc}\frac{\Omega(x-x_k)}{\pi} \quad (k = 0, 1, \cdots, 4),$$

where

$$\{x_0, x_1, x_2, x_3, x_4\} = \{-\frac{3\pi}{\Omega}, -\frac{\pi}{\Omega}, \frac{3\pi}{\Omega}, 0, \frac{4\pi}{\Omega}\}.$$

Sample points are $\{x_0, x_1, x_2\}$. Let the noise covariance be the 3×3 matrix whose elements are all $\sigma^2 = 0.09$. Then, L_n is given as

$$L_n = \mathrm{span}\{\varphi_1 + \varphi_2 + \varphi_3\}.$$

One choice of L can be

$$L_1 = \mathrm{span}\{\varphi_0 + \varphi_3 + \varphi_4,\ \varphi_1 - \varphi_3,\ \varphi_2 - \varphi_4\},$$

which satisfies $L_1 \supset L_n$ so that a signal of minimum variance is reconstructed. Another choice can be

$$L_2 = \mathrm{span}\{\varphi_1 + \hat{\varphi},\ \varphi_2 + \hat{\varphi},\ \varphi_3 + \hat{\varphi}\},$$

where $\hat{\varphi} = 3\varphi_3 + \varphi_4$. This does not have L_n as its subspace, thus minimum varianceness does not hold.

Let the original signal $f(x)$ be

$$f(x) = 5\varphi_0(x) - 4\varphi_1(x) + 2\varphi_2(x) + 19/2\varphi_3(x) + 7/2\varphi_4(x).$$

This signal was chosen so that its projections onto L_1 along $V_r \cap V_s^{\perp}$ agrees with that onto L_2 to be able to compare the reconstructed signal easily.

Fig. 1 (a) and (b) show thirty signals reconstructed from samples provided thirty times by the proposed consistent sampling theorems with L_1 and L_2, respectively. The original signal $f(x)$ and its projection $(Pf)(x)$ are also shown by the solid and dashed lines, respectively. We can clearly see that the minimum variance sampling theorem provides much stabler results than the non-minimum variance one.

V. Conclusion

This paper proposed a sampling theorem which reconstructs a consistent signal from noisy under-determined samples. When the subspace L is arbitrarily fixed, reconstructed signals are distributed in L. Thus, under the condition that the average agrees with the noiseless reconstruction, we reconstructed the signal so that the variance is minimized. Since this minimum value depends on L, we determined L so that it is further minimized. We showed that such an L can be chosen if and only if L includes a subspace L_n which is determined by the noise covariance matrix. By computer simulations, we showed the difference between the minimum variance reconstruction and the non-minimum case.

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