Modeling Image Sparsity in Compressive Sensing

Shanzhen Lan Pin Xu, Qi Zhang
Communication University of China, Beijing
E-mail: {lanshanzhen,xupin,disp}@cuc.edu.cn

***Abstract*— Compressive sensing is an emerging technology which can recover a sparse signal vector of dimension *N* via a much smaller number of non-adaptive, linear measurements than *N*. It is stated that the *K-sparse* signal *x* can be recovered exactly from $M = O(K\log(N/K))$ measurements provided the measurement matrix *A* satisfies the so-called *restricted isometry property* (RIP). However, for the compressible signal, such as the image, which is not *K*-sparse, how many measurements it requires to achieve an acceptable visual quality? In this paper, we study the relationship between the image complexity and the required measurements in compressive sensing. We propose a mathematical model based on image texture and edge density to estimate the number of needed measurements. The experimental results with a large number of natural images shows that, quite most reconstructed images using our pre-calculated number of measurements have good enough quality (PSNR > 32dB), which confirms our proposed complexity-based model well.**

I. Introduction

The conventional approach of reconstructing signals from measured data follows the well-known Shannon sampling theorem, which states that the sampling rate must be twice the highest frequency. Similarly, the fundamental theorem of linear algebra suggests that the number of collected measurements of a discrete finite-dimensional signal should be at least as large as its dimension in order to ensure reconstruction. However, the novel theory of compressive sensing (CS) provides an alternative to Shannon/Nyquist sampling when the signal under acquisition is known to be sparse or compressible [1], [2], [3]. It states that if a signal is sparse, then under certain conditions it can be reconstructed exactly from a small set of non-adaptive, linear measurements using tractable optimization algorithms.

In CS, we measure not periodic signal samples but rather inner products with $M \ll N$ measurement vectors. In matrix notation, the measurements $y = \Phi x$, where the rows of the $M \times N$ matrix Φ contain the measurement vectors. While the matrix is rank deficient, it loses information in general. In their works [4], Candès, Romberg and Tao prove that if the matrix satisfies the *restricted isometry property* (RIP), which is initially called the *uniform uncertainty principle* by them; it can preserve the information in sparse and compressible signals. A large class of random matrices has the RIP with overwhelming probability, such as Gaussian, Bernoulli, Rademacher (± 1), partial random Fourier matrices, etc. To recover the signal from the compressive measurements *y*, we search for the sparsest coefficient vector θ that agrees with the measurements. To date, researches in CS have focused primarily both on reducing the number of measurements *M* (as a function of *N* and *K*) and on increasing the robustness and reducing the computational complexity of the recovery algorithm. Today's state-of-the-art CS systems can robustly recover *K*-sparse and compressible signals from just noisy measurements using polynomial-time optimization solvers or greedy algorithms. Several introductory texts about compressive sensing, as well as a lot of reference materials can be found on [5].

In recent years, there have been growing amounts of interest in applying the results from the field of CS to imaging applications, an area known as compressive imaging. It is proved that CS is very effective in imaging [6], [7]. However, for compressible signals or images, the sparsity *K* is unknown, so the value of *M* is also undetermined. How many compressive measurements it requires to achieve an acceptable visual quality for a compressible image? To the best of our knowledge, this problem is substantially unexplored.

The sparsity is obviously relevant to the complexity of the image content. There are a wide variety of definitions for image complexity depending on its application. For example, in [8], image complexity is related to the number of objects and segments in image. Some works have related image complexity to entropy of image intensity [9]. In [10], complexity has been considered as a subjective characteristic that is represented by a fuzzy interpretation of edges in an image. These definitions clarify that there are different approaches for calculating image complexity depending on the application. Since each definition, based on either subjective or objective characteristics of the input image, uses a distinct measurement or calculation algorithm, therefore, there is not any agreement on image complexity definition. There are several image complexity measure approaches, such as Quad Tree method [11] and Image Compositional Complexity (ICC) [9].

In this paper, we firstly study the image complexity measure approaches, and propose using the texture and the edge density as the metrics of image complexity. We propound a mathematic model based on image complexity to estimate the number of required compressive measurements for the agreed reconstructed quality. With the training image set, we fit the function of the number of compressive measurements and the image complexity. And then we verify the effectiveness of the model with a large number of natural images from the test image set. The experimental results show that the PSNR of about 90% reconstructed images using our pre-calculated number of measurements is more than 32dB, which confirms our proposed complexity-based model well.

The rest of paper is organized as follows. In Section II we introduce the background of compressive sensing. In Section III we firstly describe our image complexity measure approach by the texture and the edginess and then present our proposed mathematical model based on the complexity. Section IV presents the experiments and results. Finally conclusions are provided in Section V.

II. BACKGROUND OF COMPRESSIVE SENSING

Consider a signal $x \in \mathbb{R}^{M \times N}$, which is K-sparse in an orthonormal basis Ψ with size $N \times N$; that is, $\theta \in \mathbb{R}^{N}$ defined as $\theta = \Psi^{T} x$, has at most K nonzero components. Compressive sensing [1-4] deal with the recovery of x from undersampled linear measurements of the form:

$$y = \Phi x = \Phi \Psi \theta = A\theta, \tag{1}$$

where y is a $M \times 1$ vector, $\Phi \in \mathbb{R}^{M \times N}$ is the measurement matrix that is incoherent with Ψ, and $A = \Phi\Psi$. More specifically, the M measurements in y are random linear combinations of the entries of θ, which can be viewed as the compressed and encrypted version of x. For $M < N$, estimating x from the measurements y is an ill-conditioned problem. Exploiting the sparsity of θ, CS states that the signal x can be recovered exactly from

$$M = \mathrm{O}(K \log(N / K)) \tag{2}$$

measurements provided the matrix A satisfies the so-called *restricted isometry property* (RIP). It has been shown that we can recover θ (or equivalently, x) exactly by solving the following l_0-norm minimization problem:

$$\min \|\theta\|_0 \; s.t. y = \Phi x = A\theta. \tag{3}$$

Unfortunately, it is a combinatorial, NP-hard problem; furthermore, the recovery is not stable in the presence of noise [3]. Stable recovery algorithms actually rely on the RIP. They can be grouped into two camps. The first approach convexifies the l_0-norm minimization (3) to the l_1-norm minimization

$$\min \|\theta\|_1 \; s.t. y = \Phi x = A\theta. \tag{4}$$

It corresponds to a linear program that can be solved in polynomial time. Many algorithms have been proposed to solve the convex optimization problem, including interior-point methods, projected gradient methods, and iterative thresholding.

The second approach finds the sparsest x agreeing with the measurements y through an iterative, greedy search. Algorithms such as matching pursuit, orthogonal matching pursuit, StOMP [12], CoSaMP [13], and Subspace Pursuit all build up an approximation one step at a time by making locally optimal choices at each step.

III. COMPRESSIVE SENSING BASED IMAGE COMPLEXITY

In CS we do not acquire x directly but rather acquire $M < N$ linear measurements $y = \Phi x$ using a $M \times N$ measurement matrix Φ. Then we recover x by exploiting its sparsity or compressibility. Candès demonstrate that if the measurement matrix Φ satisfies the RIP [1], the good estimate of x can be recovered. For K-sparse signals, random matrices whose entries are independent and identically distributed (i.i.d.) Gaussian, Bernoulli, Rademacher (± 1), or more generally subgaussian work with high probability provided $M = \mathrm{O}(K \log(N / K))$. For compressible signal, however, it is difficult to determine the precise number of required measurements to recover the original image with an acceptable quality. Moreover, adaptivity is crucial to capture the regularity of complex natural images.

In this paper, we propose a framework to estimate the required number M of compressive measurements based on the image complexity. Here we concern the overall description of image complexity so as to have a global grasp of image data, but not other detail messages, such as the number of objects and segments in image. Although there is not a unique method for image complexity calculation, there is a global agreement in classifying images by complexity. Among all image features, the texture and the edginess are the two most important ones for image visual complexity. Fig. 1 is the flow chart of our proposed method. In the next section, we firstly introduce our complexity measure approach, including texture metric and edginess metric, and then propose our model to estimate the number of measurements of compressive sensing for a good enough reconstruction.

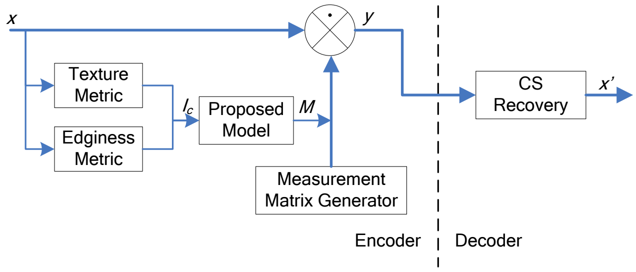

Fig. 1 The flow chart of proposed method

A. *Texture metric*

To date, many texture measures have been developed. Gray level co-occurrence matrix (GLCM) [14] is one of the most known texture analysis methods. It estimates image properties related to second-order statistics. Each entry (i, j) in GLCM corresponds to the number of occurrences of the pair of gray levels i and j which are a distance d apart in original image. The probability of gray level i to j is defined as $p_d(i, j)$. In general, there are 4 different directions for d, shown in Fig. 2.

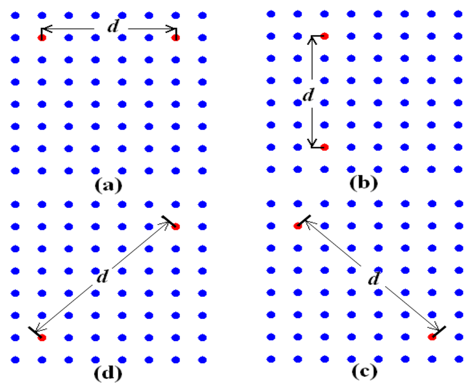

Fig. 2 Four different direction (d=5)

In order to estimate the similarity between different gray level co-occurrence matrices, Haralick [14] proposed 14 statistical features extracted from them. Among these features, the entropy measures the disorder of the image. Its mathematical equation is shown as:

$$Entropy = -\sum_{i}\sum_{j} p_d(i,j)\log p_d(i,j) \quad (5)$$

It achieves the largest value when all elements in GLCM matrix are equal, which implies it is a completely random image. When the image is texturally uniform, only few GLCM elements are large values, others are zero, which implies that entropy is very small. The entropy gives us the average information or uncertainty of a random variable, which corresponds to the image complexity.

In our experiments, we firstly divide the whole image into several regions with size 16×16, and calculate each one's entropy of GLCM. At last, the average entropy is defined as the image texture complexity, as in:

$$\bar{E}_{tex} = \frac{1}{N_R}\sum_{k=1}^{N_R} Entropy_k = -\frac{1}{N_R}\sum_{k}\sum_{i}\sum_{j} p_d(i,j,k)\log p_d(i,j,k). \quad (6)$$

where N_R is the number of regions of the image, $p_d(i,j,k)$ is the probability of gray level *i* to *j* in region *k*. In our work, the distance $d = 5$ in Fig. 2 (a).

B. Edginess metric

Excepted for the texture, the edginess is also a very important component for image visual complexity. An image which contains more prominent edges looks clearly more complex. In this paper we use the edge ratio as edginess metric, defined as in:

$$R_{edge} = N_{edge\_pixel} / N_{total\_pixel}. \quad (7)$$

where N_{total_pixel} is the total number of pixels in the image and N_{edge_pixel} is the total number of edge pixels. We make use of Prewitt edge detection method with threshold 0.04. The Prewitt operator calculates the gradient of the image intensity at each point, giving the direction of the largest possible increase from light to dark and the rate of change in that direction. Its result therefore shows how "abruptly" or "smoothly" the image changes at that point. It is exactly corresponding to the image complexity or sparsity.

C. Compressive sensing based image complexity

Next, we introduce our proposed mathematical model, which describes the relationship between the image complexity and the number of needed measurements in compressive sensing for a certain extent quality. From the above, we measure the image complexity I_c with the sum of the texture and the edginess of the image, as in

$$I_c = \bar{E}_{tex} + R_{edge}, \quad (8)$$

where $\bar{E}_{tex}$ is calculated with (6), R_{edge} is carried out with (7), and they are normalized in the range $(0,1)$, respectively. Furthermore, we do experiments to recovery the image with different M compressive measurements using Romberg's recovery algorithm in [6]. The reconstruction performance is measured by PSNR (dB). From the experiment results, we estimate the required M value for PSNR=32 dB of the reconstructed image. Our training image set includes 100 images downloaded from USC SIPI Image database. Fig. 3 presents the image sparsity *vs* the complexity for training images. And then we fit the experimented data with least-

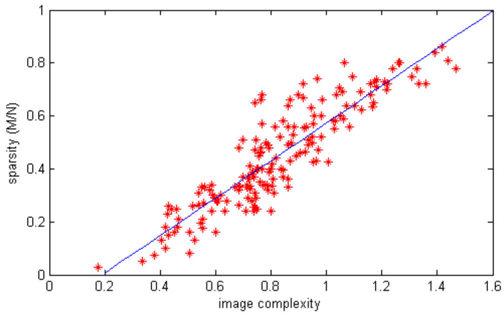

Fig. 3 The sparsity *vs* the complexity for training images. The line is the fitted result with least-squares approximation.

squares approximation to a linear function, as the line in Fig.3. The fitted result is formulated as:

$$M = f(I_c) = \alpha \cdot I_c + \beta, \quad (9)$$

where M is the estimated number of compressive measurements, I_c is the image complexity using (8), α and β are fitting coefficients. In our experiments, $\alpha = 0.70$, $\beta = -0.16$. In Section IV we verify the model with a large number of test images.

IV. EXPERIMENTS AND RESULTS

The experiments are conducted as follows. Firstly, the test image set contains 5000 images in good quality. 3000 of them are downloaded from the USDA NRCS Photo Gallery [1]. Another 2000 images are collected from several types of digital cameras. All images are resampled to make all the images in the size 256×256 and converted into gray-scale. We use 800 different images, which chosen randomly from the above set, some of them listed in Fig. 4, and calculate the

Fig. 4 Several images from Test Image Set.

[1] http://photogallery.nrcs.usda.gov

required number of compressive measurements for PSNR=32 dB reconstructed quality using (9). Then the images are reconstructed with the estimated number of measurements using Romberg's recovery algorithm [6].

Fig. 5 presents the result of compressive sensing recovery for 800 images in Test Image Set by taking an estimated number of measurements of the images by (9). It is measured by PSNR (dB). It is shown that most of images achieve good enough visual quality. Fig. 6 shows the cumulative distribution function (CDF) of the reconstructed quality. It is clearly seen that about 90% images have more than 32 dB visual quality. The result demonstrates that our proposed mathematical model is the workable and suitable for compressive sensing.

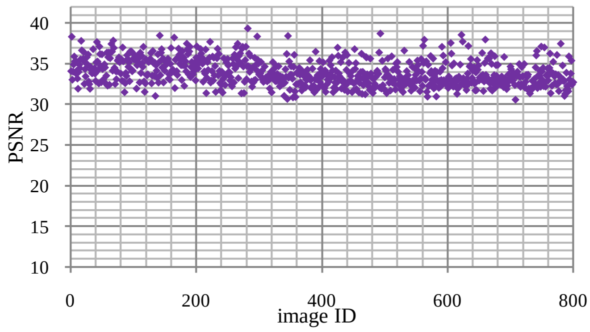

Fig. 5 PSNR of the reconstructed image.

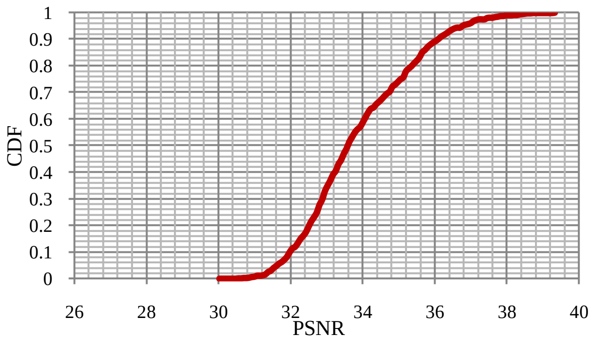

Fig. 6 CDF of 800 images recovery PSNR using proposed estimated *M*.

V. CONCLUSIONS

Compressive sensing (CS) is a new research topic in signal processing that has promoted the interest of a wide range of researchers in different fields recently. In this paper, we propose a complexity-based mathematic model to estimate the number of required measurements for compressive imaging. Among lots of image features, we use the texture and the edginess, which are the most two important ones to its visual complexity as our complexity measure. The experimental results with a large number of real-world images shows that, more than 90% reconstructed images have good enough quality (PSNR > 32dB) when taken our pre-calculated number of measurements. Thus, it is confirmed that our proposed complexity-based model performs well.

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