Low-complexity approximate LMMSE channel estimation for OFDM systems

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***Abstract*—A low-complexity linear minimum mean square error (LMMSE) based channel estimator is proposed for orthogonal frequency-division multiplexing (OFDM) systems over frequency-selective channels. Using the law of a large number, we approximate the LMMSE estimator to reduce the numerical complexity of the channel estimation. Our estimator exhibits comparable performance with the LMMSE estimator at low SNR but suffers the performance floor due to the approximation, which are verified by numerical simulations.**

I. INTRODUCTION

Orthogonal Frequency Division Multiplexing (OFDM) [1] has been applied in many wireless communications systems due to its high rate transmission capability together with high bandwidth efficiency as well as its robustness with regard to multi-path fading and delay. These advantages are achieved by transmitting the redundant signal known as cyclic prefix.

In general, channel estimation at the receiver is required for coherent decoding. Since the estimated channel is used, the accuracy of the channel estimators strongly affects the performance of the receiver. This also holds true for OFDM systems. In OFDM systems, the estimation of frequency selective channels is necessary, since an OFDM system occupies a relatively large bandwidth. Several channel estimation methods have been proposed for OFDM systems, see e.g. [2] and references therein.

OFDM preambles, which are known at the receiver, are transmitted to obtain the channel state information (CSI) at the beginning of the transmission record. Optimal preambles for channel estimation have been designed. If all subcarriers are available for channel estimation, then symbols of the optimal preamble have the equal power, which facilitate channel estimation. However, if there are some null subcarriers, then symbols of the optimal preamble do not have equal power except for some special cases. If the symbols of the preamble have unequal powers, then $O(N^3)$ computations are required to obtain the linear minimum mean square error (LMMSE) channel estimate, where N is the number of subcarriers.

CSI by preamble will be outdated during the data transmission if channels are changing in time. To enable channel estimation, pilot symbols, which are known at the receiver, are transmitted on certain subcarriers of each OFDM symbol. This is called pilot symbol-assisted modulation (PSAM) [3], [4]. The design of pilot symbols are studied e.g. in [5]. One major drawback of PSAM is the reduction of bandwidth efficiency, since the number of known symbols has to be comparable with the number of channel coefficients. On the other hand, decoded information symbols can be utilized to estimate channel, which is known as decision-direct channel estimation. The iterative signal detection and channel estimation for OFDM systems have been also studied. If information symbols do not have the equal power, that is, information symbols are not drawn from PSK constellations, then it necessitates $O(N^3)$ computations to compute an LMMSE channel estimate.

In [6], low-complexity LMMSE based estimators have been proposed by using the frequency correlation matrix of the channel and its singular value decomposition (SVD). Thanks to the SVD, once the estimator is given, the complexity of the estimator is small. However, to obtain the estimator, an SVD of an $N \times N$ matrix has to be computed, which requires $O(N^3)$ computations. To avoid the SVD, robust channel estimation have been developed for constant-modulus signaling [7], which utilizes pre-calculated SVD of typical channels.

This paper proposes low-complexity LMMSE based estimators for general non-constant-modulus signaling. Using the law of a large number, we approximate a matrix in the LMMSE estimator to be an identity matrix, which significantly reduces the numerical complexity of the channel estimation. Ignoring the correlation between different channel coefficients, we further approximate the channel covariance matrix as a diagonal matrix. Then, the numerical computations of our channel estimator can be reduced to be $O(N \log_2 N)$ when N is a power of 2. Numerical simulations are provided to show that our low-complexity estimator exhibits comparable performance with the theoretical LMMSE estimator at low SNR but suffers the performance floor due to the approximation.

II. CHANNEL ESTIMATION IN OFDM SYSTEMS

Let us consider a point-to-point wireless OFDM system with N subcarriers over frequency selective channels. The discrete-time baseband equivalent channel is assumed to be time-invariant at least for one OFDM symbol duration and to have coefficients $h_0, \ldots, h_{L-1}$ of maximum length L. We define the channel frequency response at frequency $2\pi k/N$ as

$$H_k = \sum_{l=0}^{L-1} h_l e^{-j\frac{2\pi kl}{N}}, \text{ for } k \in [0, N-1]. \tag{1}$$

At the transmitter, a symbol sequence $\{s_0, s_1, \ldots, s_{N-1}\}$ undergoes serial-to-parallel (S/P) conversion to be stacked into one OFDM symbol. Then, an N-points inverse discrete Fourier transform (IDFT) follows to produce N dimensional data, which is parallel-to-serial (P/S) converted.

Prior to transmission, a guard interval also known as a cyclic prefix (CP) is appended to each OFDM symbol to mitigate multipath effects. We assume that the length of the cyclic prefix N_{cp}, is greater than the channel length L so that there is no inter-symbol interference (ISI) between consecutive OFDM symbols.

At the receiver, we assume perfect timing and frequency synchronization. After removing CP, we apply discrete Fourier transform (DFT) to the received time-domain signal to obtain the baseband signal at the kth subcarrier, which can be expressed as [1]

$$Y_k = H_k s_k + W_k, \tag{2}$$

for $k \in [0, N-1]$, where s_k is the transmitted symbol at kth subcarrier and W_k is assumed to be i.i.d. circular Gaussian with zero mean and variance σ_w^2.

In OFDM systems, OFDM preambles, which are known at the receiver, are transmitted to obtain the channel state information (CSI) at the beginning of the transmission record. We can estimate H_k from (2) since s_k is known. On the other hand, in decision-direct mode, s_k is an unknown information symbol to be detected by using the estimated channel from the previous OFDM symbol. Then, the channel estimate can be updated from the detected s_k. This paper considers both cases, assuming that s_k is known at the receiver.

In practice, not all the subcarriers are used for transmission. It is often the case that null subcarriers are set on both edges of the allocated bandwidth to mitigate interferences from/to adjacent bands [8]. For example, IEEE 802.11a has 64 subcarriers among which 12 subcarriers, one at the center of the band (DC component) and at the edges of the band are set to be nulled, i.e., no information is sent [1]. At the null subcarriers, we just set $s_k = 0$. We also assume that the number of active subcarriers are not less than L, which implies that if there is no noise, the channel can be obtained from the received signals.

Let us define an $N \times N$ DFT matrix as $\boldsymbol{F}$, whose $(m+1, n+1)$th entry is $e^{-j2\pi mn/N}$. We denote an $N \times L$ matrix as $\boldsymbol{F}_L$, which consists of N rows and first L columns of DFT matrix $\boldsymbol{F}$.

Let $\mathrm{diag}(\boldsymbol{a})$ be a diagonal matrix with the vector $\boldsymbol{a}$ on its main diagonal. We collect the received signals as

$$\boldsymbol{Y} = [Y_0, \ldots, Y_{N-1}]^T. \tag{3}$$

It follows from (2) that

$$\boldsymbol{Y} = \boldsymbol{D}_s \boldsymbol{F}_L \boldsymbol{h} + \boldsymbol{W}, \tag{4}$$

where $\boldsymbol{W} = [W_0, \ldots, W_{N-1}]$ is the noise vector, the diagonal matrix $\boldsymbol{D}_s$ and the channel vector $\boldsymbol{h}$ are respectively defined as

$$\boldsymbol{D}_s = \mathrm{diag}(s_0, s_1, \ldots, s_{N-1}) \tag{5}$$

and

$$\boldsymbol{h} = [h_0, \ldots, h_{L-1}]^T. \tag{6}$$

It should be remarked that the frequency-domain channel

$$\boldsymbol{H} = [H_0, H_1, \ldots, H_{N-1}]^T \tag{7}$$

is related to the time-domain channel $\boldsymbol{h}$ by

$$\boldsymbol{H} = \boldsymbol{F}_L \boldsymbol{h}. \tag{8}$$

We denote the expectation operator as $E\{\cdot\}$. Assuming that $E\{\boldsymbol{h}\} = 0$, we express the covariance matrix of $\boldsymbol{h}$ as

$$\boldsymbol{R}_h = E\{\boldsymbol{h}\boldsymbol{h}^{\mathcal{H}}\}, \tag{9}$$

where $\mathcal{H}$ is the complex conjugate transpose operator. We also assume that transmitted symbols are statistically independent of the noises. Since the transmitted symbols are uncorrelated to the noises, it is easy to see that

$$\boldsymbol{C}_{YY} = E\{\boldsymbol{Y}\boldsymbol{Y}^{\mathcal{H}}\} = \boldsymbol{D}_s \boldsymbol{F}_L \boldsymbol{R}_h \boldsymbol{F}_L^{\mathcal{H}} \boldsymbol{D}_s^{\mathcal{H}} + \sigma_w^2 \boldsymbol{I}_N, \tag{10}$$

where $\boldsymbol{I}_N$ is the identity matrix of size $N \times N$. Similarly, we have

$$\boldsymbol{C}_{HY} = E\{\boldsymbol{H}\boldsymbol{Y}^{\mathcal{H}}\} = \boldsymbol{F}_L \boldsymbol{R}_h \boldsymbol{F}_L^{\mathcal{H}} \boldsymbol{D}_s^{\mathcal{H}}. \tag{11}$$

From these, the LMMSE estimate of the frequency-domain channel is given by

$$\hat{\boldsymbol{H}} = \boldsymbol{C}_{HY} \boldsymbol{C}_{YY}^{-1} \boldsymbol{Y}. \tag{12}$$

Except for some special cases, $\boldsymbol{C}_{YY}$ is not a diagonal matrix. Thus, in general, it requires $O(N^3)$ computations to obtain the inverse of $\boldsymbol{C}_{YY}$, which makes it difficult to utilize the LMMSE estimator on some devices. In this paper, we develop an approximate LMMSE estimator which can be computed with much less computations.

III. Approximate LMMSE channel estimator

For the simplicity of presentation, let us assume that the covariance matrix $\boldsymbol{R}_h$ of $\boldsymbol{h}$ is a positive definite matrix, which can be diagonalized with a unitary matrix $\boldsymbol{U}$ and a diagonal matrix

$$\boldsymbol{\Lambda}_h = \mathrm{diag}(\lambda_0, \lambda_1, \ldots, \lambda_{L-1}) \tag{13}$$

such that

$$\boldsymbol{R}_h = \boldsymbol{U} \boldsymbol{\Lambda}_h \boldsymbol{U}^{\mathcal{H}}. \tag{14}$$

We define the transmit power as

$$\mathcal{P} = \mathrm{trace}(\boldsymbol{D}_s \boldsymbol{D}_s^{\mathcal{H}}). \tag{15}$$

Now, applying the matrix inversion lemma to $\boldsymbol{C}_{YY}$, we obtain

$$\begin{aligned} \boldsymbol{C}_{YY}^{-1} &= \sigma_w^{-2} \boldsymbol{I} - \\ &\sigma_w^{-4} \boldsymbol{D}_s \boldsymbol{F}_L (\boldsymbol{R}_h^{-1} + \sigma_w^{-2} \boldsymbol{F}_L^{\mathcal{H}} \boldsymbol{D}_s^{\mathcal{H}} \boldsymbol{D}_s \boldsymbol{F}_L)^{-1} \boldsymbol{F}_L^{\mathcal{H}} \boldsymbol{D}_s^{\mathcal{H}} \end{aligned} \tag{16}$$

During the decision directed mode, the entries of $\boldsymbol{s}$ are randomly generated digital symbols except for pilot symbols. Then, from the law of large numbers, we can use the approximation:

$$\boldsymbol{F}_L^{\mathcal{H}} \boldsymbol{D}_s^{\mathcal{H}} \boldsymbol{D}_s \boldsymbol{F}_L \approx \mathcal{P} \boldsymbol{I} \tag{17}$$

if the number of active subcarriers are large enough. If s_k has constant-modules, then we have

$$\boldsymbol{D}_s^{\mathcal{H}}\boldsymbol{D}_s = c\boldsymbol{I} \tag{18}$$

with a constant $c > 0$. Thus, the equality holds in (17).

On the other hand, a pseudo-random sequence can be utilized for the preamble OFDM symbol. Therefore, although the preamble OFDM symbol is deterministic, the approximation is valid also for the well-designed preamble OFDM symbol.

It follows from (11) and (16) that

$$\begin{aligned} & \boldsymbol{C}_{HY}\boldsymbol{C}_{YY}^{-1} \\ = \; & \sigma_w^{-2}\boldsymbol{F}_L\boldsymbol{R}_h\boldsymbol{F}_L^{\mathcal{H}}\boldsymbol{D}_s^{\mathcal{H}} \cdot \\ & \left[\boldsymbol{I}_N - \sigma_w^{-2}\boldsymbol{D}_s\boldsymbol{F}_L\left(\boldsymbol{R}_h^{-1} + \sigma_w^{-2}\mathcal{P}\boldsymbol{I}_L\right)^{-1}\boldsymbol{F}_L^{\mathcal{H}}\boldsymbol{D}_s^{\mathcal{H}}\right] \\ = \; & \sigma_w^{-2}\boldsymbol{F}_L\boldsymbol{R}_h \cdot \\ & \left[\boldsymbol{I}_L - \sigma_w^{-2}\boldsymbol{F}_L^{\mathcal{H}}\boldsymbol{D}_s^{\mathcal{H}}\boldsymbol{D}_s\boldsymbol{F}_L\left(\boldsymbol{R}_h^{-1} + \sigma_w^{-2}\mathcal{P}\boldsymbol{I}_L\right)^{-1}\right]\boldsymbol{F}_L^{\mathcal{H}}\boldsymbol{D}_s^{\mathcal{H}} \end{aligned}$$

Using the approximation (17) again in the above equation leads to

$$\begin{aligned} & \boldsymbol{C}_{HY}\boldsymbol{C}_{YY}^{-1} \\ = \; & \sigma_w^{-2}\boldsymbol{F}_L\boldsymbol{R}_h\left[\boldsymbol{I}_L - \sigma_w^{-2}\mathcal{P}\left(\boldsymbol{R}_h^{-1} + \sigma_w^{-2}\mathcal{P}\boldsymbol{I}_L\right)^{-1}\right]\boldsymbol{F}_L^{\mathcal{H}}\boldsymbol{D}_s^{\mathcal{H}}. \end{aligned} \tag{19}$$

Using $\boldsymbol{R}_h = \boldsymbol{U}\boldsymbol{\Lambda}_h\boldsymbol{U}^{\mathcal{H}}$ and $\boldsymbol{R}_h^{-1} = \boldsymbol{U}\boldsymbol{\Lambda}_h^{-1}\boldsymbol{U}^{\mathcal{H}}$, we have

$$\begin{aligned} & \sigma_w^{-2}\boldsymbol{R}_h\left[\boldsymbol{I}_L - \sigma_w^{-2}\mathcal{P}\left(\boldsymbol{R}_h^{-1} + \sigma_w^{-2}\mathcal{P}\boldsymbol{I}_L\right)^{-1}\right] \\ = \; & \sigma_w^{-2}\boldsymbol{U}\boldsymbol{\Lambda}_h\left[\boldsymbol{I}_L - \sigma_w^{-2}\mathcal{P}\left(\boldsymbol{\Lambda}_h^{-1} + \sigma_w^{-2}\mathcal{P}\boldsymbol{I}_L\right)^{-1}\right]\boldsymbol{U}^{\mathcal{H}} \\ = \; & \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{U}^{\mathcal{H}} \end{aligned} \tag{20}$$

where

$$\boldsymbol{\Sigma} = \operatorname{diag}\left(\frac{\lambda_0}{\mathcal{P}\lambda_0 + \sigma_w^2}, \ldots, \frac{\lambda_{L-1}}{\mathcal{P}\lambda_{L-1} + \sigma_w^2}\right). \tag{21}$$

From (20), we finally obtain

$$\boldsymbol{C}_{HY}\boldsymbol{C}_{YY}^{-1} = \boldsymbol{F}_L\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{U}^{\mathcal{H}}\boldsymbol{F}_L^{\mathcal{H}}\boldsymbol{D}_s^{\mathcal{H}}. \tag{22}$$

It should be noted that the low complexity estimator in [6] has been derived by replacing $(\boldsymbol{D}_s\boldsymbol{D}_s^{\mathcal{H}})^{-1}$ by $E\{1/|s_k|^2\}\boldsymbol{I}_N$ which does not hold true even asymptotically as $N \to \infty$ except for constant-modulus signals, while our approximation does asymptotically.

Our estimator (22) requires the eigenvalue decomposition of the covariance matrix $\boldsymbol{R}_h$ of size $L \times L$, whose complexity is $O(L^3)$. Since L is much smaller than N, the complexity of the eigenvalue decomposition is not so heavy. However, it may be better to avoid the eigenvalue decomposition.

To further reduce the complexity, we ignore the correlation between different time-domain channel coefficients. That is, we approximate $\boldsymbol{R}_h$ to be diagonal such that

$$\boldsymbol{R}_h \approx \operatorname{diag}(\sigma_{h_0}^2, \sigma_{h_1}^2, \ldots, \sigma_{h_{L-1}}^2) \tag{23}$$

where

$$\sigma_{h_l}^2 = E\{|h_l|^2\}. \tag{24}$$

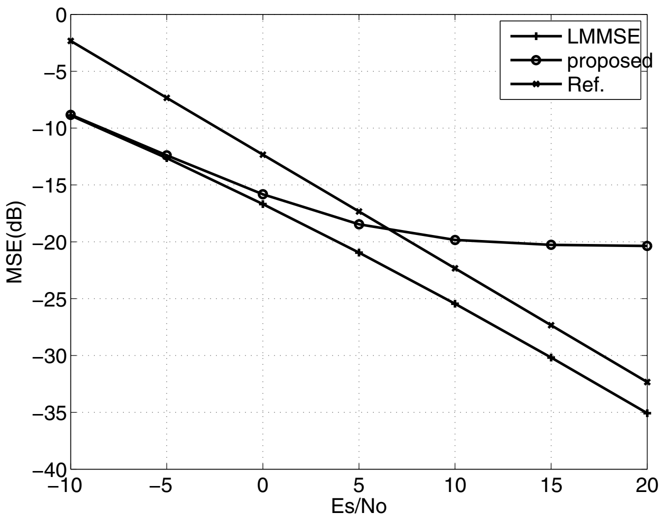

Fig. 1. MSE of LMMSE estimator (with $+$), approximate estimator (with $\circ$), and referenced estimator (with x) for $\alpha = 0$).

Setting $\boldsymbol{U} = \boldsymbol{I}_L$ and $\lambda_l = \sigma_{h_l}^2$ for $l \in [0, L-1]$ in (25), we have

$$\boldsymbol{C}_{HY}\boldsymbol{C}_{YY}^{-1} = \boldsymbol{F}_L\boldsymbol{\Sigma}\boldsymbol{F}_L^{\mathcal{H}}\boldsymbol{D}_s^{\mathcal{H}}. \tag{25}$$

Then the estimate of $\boldsymbol{H}$ is given by

$$\hat{\boldsymbol{H}} = \boldsymbol{F}_L\boldsymbol{\Sigma}\boldsymbol{F}_L^{\mathcal{H}}\boldsymbol{D}_s^{\mathcal{H}}\boldsymbol{Y}, \tag{26}$$

which can be obtained by multiplications of diagonal matrices $\boldsymbol{\Sigma}$ and $\boldsymbol{D}_s^{\mathcal{H}}$, one DFT with $\boldsymbol{F}_L$ and one IDFT with $\boldsymbol{F}_L^{\mathcal{H}}$. Thus, the computational complexity is $O(N \log_2 N)$ if N is a power of 2.

IV. Simulation Results

To access the performance of the proposed estimators, we utilize OFDM systems with subcarriers $N = 256$ and CP length $N_{cp} = 16$, where all subcarriers are available, i.e., there is no null subcarrier.

The channel length L is 8. The channel has the exponential power profile given by $\sigma_{h_l}^2 = e^{-l}/\sum_l e^{-l}$. It is noted that there is no ISI since CP is sufficiently long.

Channel coefficients are Gaussian distributed. We consider two cases: uncorrelated channel coefficients and correlated channel coefficients. We control the correlation using a parameter α. For correlated channel coefficients, the off-diagonal entries of the correlation matrix are set to be

$$[\boldsymbol{R}_h]_{mn} = \alpha\sigma_{h_m}\sigma_{h_n}, \tag{27}$$

where $[\cdot]_{mn}$ is the (m, n)th entry of the matrix. This implies that the normalized correlation between the mth and the nth channel coefficient is α.

The transmitted signals are drawn from the 16QAM constellation. In this case, strictly speaking, the equality does not hold in (17). Thus, our estimator can be considered as an

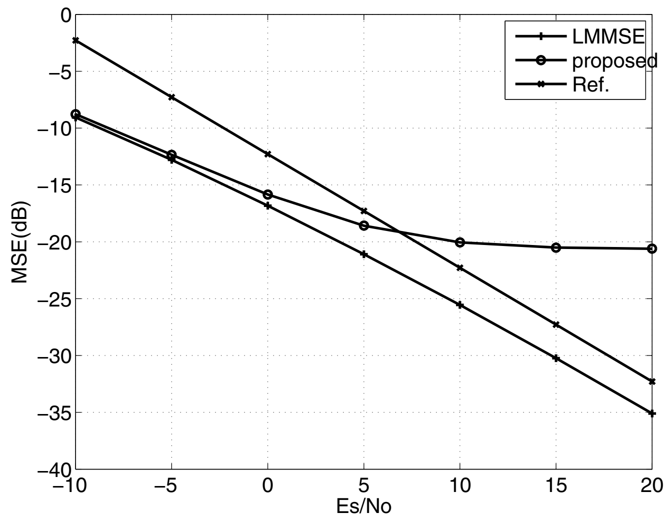

Fig. 2. MSE of LMMSE estimator (with $+$), approximate estimator (with $\circ$), and referenced estimator (with x) for $\alpha = 0.3$).

approximate LMMSE estimator even for uncorrelated channel coefficients. We define the average MSE as

$$\frac{1}{N}E\{||\boldsymbol{H} - \hat{\boldsymbol{H}}||^2\}. \tag{28}$$

We generate 10^3 channels and average the MSE for every channel.

For $\alpha = 0$, Fig. 1 compares the MSE of our approximate estimator with the MSE of the LMMSE estimator as well as the MSE of the estimator proposed in [9], where E_s is the energy per symbol given by $E_s = E\{|s_k|^2\}$ and $N_0 = \sigma_w^2$. The channel estimate of [9] is computed by

$$\boldsymbol{F}_L \boldsymbol{F}_L^{\mathcal{H}} \boldsymbol{D}_s^{-1} \boldsymbol{Y}, \tag{29}$$

with computational complexity is also of $O(N \log_2 N)$.

At low SNR, our estimator exhibits comparable performance with the LMMSE estimator and better performance than the referenced estimator. However, our estimator suffers a performance floor. This may be due to the fact that the approximation (17) is independent of the noise variance and hence even at high SNR our estimator is affected by the approximation errors.

Fig. 2 depicts the MSE for $\alpha = 0.3$. Each estimator has similar MSE with the corresponding MSE for $\alpha = 0$, which suggests that small correlations between channel coefficients do not affect the channel estimate MSE. Again, our estimator has the performance floor due to the approximation errors.

V. Conclusions

We have developed a low-complexity channel estimator for OFDM systems over frequency-selective channels by using the law of a large number. Our estimator exhibits comparable performance with the LMMSE estimator at low SNR but suffers from the performance floor due to the approximation.

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