Some Problems in Demand Side Management

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Abstract—We present a sample of problems in demand side management in future power systems and illustrate how they can be solved in a distributed manner using local information. First, we consider a set of users served by a single load-serving entity (LSE). The LSE procures capacity a day ahead. When random renewable energy is realized at delivery time, it manages user load through real-time demand response and purchases balancing power on the spot market to meet the aggregate demand. Hence optimal supply procurement by the LSE and the consumption decisions by the users must be coordinated over two timescales, a day ahead and in real time, in the presence of supply uncertainty. Moreover, they must be computed jointly by the LSE and the users since the necessary information is distributed among them. We present distributed algorithms to maximize expected social welfare. Instead of social welfare, the second problem is to coordinate electric vehicle charging to fill the valleys in aggregate electric demand profile, or track a given desired profile. We present synchronous and asynchronous algorithms in aggregate electric demand profile, or track a given desired expected social welfare. Instead of social welfare, the second problem is to coordinate electric vehicle charging to fill the valleys in aggregate electric demand profile, or track a given desired profile. We present synchronous and asynchronous algorithms and prove their convergence. Finally, we show how loads can be solved in a distributed manner using local information. This entitled the LSE to purchase the amount of the day-ahead energy that the LSE actually uses the following day and to minimize its total energy cost is

\[ \Delta(P_o) = \sum_{i} \Delta(P_i) \]

where \( P_i \) is the amount of energy that the LSE actually uses on the following day. The LSE procures energy for delivery in two steps. First, one day in advance, it procures “day-ahead” capacities \( P_d \) and pays \( c_o(P_d) \) for the capacity. This entitles the LSE to purchase up to \( P_d \) amount of energy the following day at a price pre-determined by the day-ahead market. Let \( P_o(t) \) denote the amount of the day-ahead energy that the LSE actually uses the following day and \( c_o(P_o) \) denote its cost. The renewable energy is a nonnegative random variable \( P_r \) and we assume its cost is zero. At real time, the random variable \( P_r \) is realized and used to satisfy demand. The LSE satisfies any excess demand by using \( P_o \) from the day-ahead capacity. If there is still excess demand, the LSE purchases the balance \( P_b \) on the real-time energy market at a cost \( c_b(P_b) \).

The real-time decisions \( (P_o, P_b) \) are made by the LSE so as to minimize its total cost. Let \( Q := \sum_{i} q_i \) be the total demand and \( \Delta(Q) := Q - P_r \) be the excess demand, i.e., in excess of the renewable generation. The excess demand \( \Delta(Q(t)) \) and the day-ahead capacity \( P_d \), the LSE’s decision that minimizes its total energy cost is

\[ P_o^* = [\Delta(Q)]_0^{P_d}, \]
\[ P_b^* = [\Delta(Q) - P_d]_+. \]

The total supply cost that the LSE incurs then is a function only of \( P_d \) and \( Q \) and given by

\[ c(Q, P_d; P_o) = c_d(P_d) + c_o([\Delta(Q)]_0^{P_d}) + c_b([\Delta(Q) - P_d]_+) , \]

i.e., the total cost consists of the capacity cost \( c_d \), the cost \( c_o \) of day-ahead energy, and the cost \( c_b \) of the real-time balancing energy.

We make the following assumptions:

A1: The utility functions \( U_i \) are strictly concave, increasing, and continuously differentiable, and the cost functions \( c_d, c_o, c_b \) are convex, increasing, and continuously differentiable, with \( c_d(0) = c_o(0) = c_b(0) = 0 \).

A2: \( c_o'(0) > c_o'(P_o) \) for all \( P_o \geq 0 \).
B. Optimal demand response

The welfare maximization reduces to the problem

$$\max_{P_d \geq 0} \left\{ -c_d(P_d) + E \max_{q \in [\bar{q}, \tilde{q}]} W_1(q; P_d, P_r) \right\},$$

(1)

where the real-time welfare, given decision $P_d$ and realization of $P_r$, is

$$W(q; P_d, P_r) := \sum_i U_i(q_i) - c_o \left( [\Delta(Q)]_{P_d}^0 \right) - c_b \left( [\Delta(Q) - P_d]_+ \right).$$

(2)

The expectation $E$ in (1) is taken with respect to $P_r$. The order of maximizations and expectation in (1) reflects the fact that the decision $P_d$ must be made a day ahead based on the distribution of $P_r$, but the consumption decisions $q$ should be made in real time after $P_r$ is realized. Given $P_d$ and a realization of $P_r$, $W(q; P_d, P_r)$ is a deterministic function of $q$. Hence our problem decomposes into two subproblems:

1) Real-time demand response: Optimize real-time welfare $W_1$ over consumptions $q$, given $P_d, P_r$:

$$\max_{q \in [\bar{q}, \tilde{q}]} W(q; P_d, P_r) = \sum_i U_i(q_i) - c_o \left( [\Delta(Q)]_{P_d}^0 \right) - c_b \left( [\Delta(Q) - P_d]_+ \right).$$

(3)

Let $q(P_d, P_r)$ denote an optimizer.

2) Day-ahead capacity procurement: maximize expected welfare over $P_d$:

$$\max_{P_d \geq 0} \{ -c_d(P_d) + EW_1(q(P_d, P_r); P_d, P_r) \}$$

We now consider each subproblem in turn.

For the real-time demand response subproblem, since $P_d$ has been committed, the cost $c_d(P_d)$ has been given. Hence, (3) is equivalent to

$$\tilde{W}(P_d; P_r) := \max_{q \in [\bar{q}, \tilde{q}]} \left\{ \sum_i \left( \delta_i U_i(q_i) - c_o(y_o) - c_b(y_b) \right) \right\}$$

s.t. $\bar{q}_i \leq q_i \leq \tilde{q}_i, \forall i$,

$0 \leq y_o \leq P_d, y_b \geq 0$,

$P_r + y_o + y_b \geq \sum_i q_i$.

Associate dual variables $\mu_1$ and $\mu_2$ with the last two constraints. Then a partial Lagrangian is

$$\mathcal{L}(q, y_o, y_b; \mu_1, \mu_2) = \sum_i U_i(q_i) - c_o(y_o) - c_b(y_b) + \mu_1(P_d - y_o)$$

$$+ \mu_2(P_r + y_o + y_b - \sum_i q_i).$$

(4)

Consequently, a primal-dual algorithm to solve problem (4) is as follows.

Algorithm 1: Given $P_d, P_r$, compute real-time consumption

Initially, user $i$ sets $q_i^0 \in [\bar{q}_i, \tilde{q}_i]$. The LSE lets $\mu_1^0 = \mu_2^0 = 0$, and $y_o^0 = y_b^0 = 0$. In iteration $k + 1 = 1, 2, \ldots$, do the following.

1. Each user $i$ computes

$$q_i^{k+1} = \left( q_i^k + \beta k \cdot \left[ u_i(x_i^k) - \mu_2^k \right] \right)^{\frac{\bar{q}_i}{\mu_1^k}},$$

where $\beta k := \frac{1}{k+1}$ is the step size, and reports it to the LSE through a communication network. That is, the “price” posed to the users is $\mu_2^k$.

2. The LSE computes

$$\mu_1^{k+1} = [\mu_1^k + \beta (y_o^k - P_d)]_{+},$$

$$\mu_2^{k+1} = [\mu_2^k + \beta (\sum_i q_i^k - P_r - y_o^k - y_b^k)]_{+},$$

$$y_o^{k+1} = [y_o^k + \beta (c_o(y_o^k) - \mu_2^k) y_o^k P_{max}],$$

$$y_b^{k+1} = [y_b^k + \beta (c_b(y_b^k) + \mu_2^k) y_b^k P_{max}],$$

where $P_{max} := \sum_i \bar{q}_i$. The LSE reports $\mu_2^{k+1}$ to active users.

It follows directly from convex optimization theory that

Theorem 1. Algorithm 1 converges to the set of optimal solutions $q^*$ and dual variables $\mu^*$.

For the day-ahead capacity procurement subproblem, to decide $P_d$ to maximize expected social welfare, the LSE solves

$$\max_{P_d \geq 0} \{ E[\tilde{W}(P_d; P_r)] - c_d(P_d) \},$$

where $\tilde{W}(.)$ is defined in (4). The gradient of the objective function is $g(P_d) := E(\tilde{W}(P_d)) - c_d(P_d)$ (note that $\mu_1^*$ depends on $P_d, P_r$). A stochastic subgradient algorithm that converges to the set of optimal $P_d$ is as follows.

Algorithm 2: Day-ahead energy

1. Initially, let $P_d^0 = 0$.

2. In step $m + 1 = 1, 2, \ldots$, given a realization of $P_r$ and $\delta$ (denoted by $P_r^m$, run Algorithm 1 to find $\mu_1^m$, and denote it by $\mu_1^{*m}$. Then, compute

$$P_d^{m+1} = \left( P_d^m + \alpha (\mu_1^{*m} - c_d(P_d^m)) \right)^{P_{max}}$$

where $\alpha^m = 1/(m + 1)$ is the step size.

Algorithm 2 can be run one day in advance by simulating the system (i.e., drawing samples of $P_r$).

Theorem 2. Algorithm 2 converges to a welfare-maximizing procurement $P_d^*$ almost surely.

III. SCHEDULING OF EV CHARGING

A. Problem Formulation

Consider a scenario where an electric utility negotiates with $N$ electric vehicles (EVs) over $T$ time slots of length $\Delta T$ on their charging profiles. The utility is assumed to know (precisely predict) the inelastic base electricity load profile (aggregated non-EV load) and aims to shape the aggregated charging profile of EVs to flatten the total load (base load plus EV load) profile. Each EV can charge after it plugs in and needs to be charged a specified amount of electricity by...
its deadline. For instance, an EV may plug in for charging at 9:00 pm, specifying that it needs to be fully charged by 6:00 am the next morning, or at least 80% full by 4:00 am the next morning. In each time slot, the charging rate of an EV is a constant. Let \( D(t) \) denote the base load in slot \( t \), \( r_n(t) \) denote the charging rate of EV \( n \) in slot \( t \), \( r_n := (r_n(1), \ldots, r_n(T)) \) denote the charging profile of EV \( n \), for \( n \in \mathcal{N} := \{1, \ldots, N\} \) and \( t \in \mathcal{T} := \{1, \ldots, T\} \). Our goal is to flatten the total load profile. This motivates the cost function

\[
L(r) = L(r_1, \ldots, r_N) := \sum_{t=1}^{T} U(D(t) + \sum_{n=1}^{N} r_n(t)).
\]  

In (5), \( r := (r_1, \ldots, r_N) \) denotes a charging profile of all EVs.

The charging profile \( r_n \) of EV \( n \) can take values in the interval \([0, \tau_n] \) for some given \( \tau_n \geq 0 \), i.e.,

\[
0 \leq r_n(t) \leq \tau_n(t), \quad n \in \mathcal{N}, \quad t \in \mathcal{T}.
\]  

In order to impose arrival time and deadline constraints, \( \tau_n(t) \) is considered to be time dependent with \( \tau_n(t) = 0 \) for slots \( t \) before the arrival time and after the deadline of EV \( n \). For each EV \( n \in \mathcal{N} \), let \( B_n, s_n(0), s_n(T) \) and \( \eta_n \) denote its battery capacity, initial state of charge, final state of charge and charging efficiency respectively. The constraint that EV \( n \) needs to reach \( s_n(T) \) state of charge by its deadline is captured by charging a pre-specified amount of energy over time,

\[
\eta_n \sum_{t \in \mathcal{T}} r_n(t) \Delta T = B_n(s_n(T) - s_n(0)), \quad n \in \mathcal{N}.
\]  

Define the charging rate sum

\[
R_n := B_n(s_n(T) - s_n(0))/\left(\eta_n \Delta T\right)
\]

for \( n \in \mathcal{N} \). Then, the constraint in (7) can be written as

\[
\sum_{t=1}^{T} r_n(t) = R_n, \quad n \in \mathcal{N}.
\]  

**Definition 1.** Let \( U : \mathbb{R} \to \mathbb{R} \) be strictly convex. A charging profile \( r = (r_1, \ldots, r_N) \) is

1. **feasible,** if it satisfies the constraints (6) and (8);
2. **optimal,** if it solves the optimal charging (OC) problem

\[
\text{OC \quad \min_{r_1, \ldots, r_N} \sum_{t=1}^{T} U(D(t) + \sum_{n=1}^{N} r_n(t)) \quad \text{s.t.} \quad \sum_{n=1}^{N} r_n(t) = R_n, \quad n \in \mathcal{N}; \quad 0 \leq r_n(t) \leq \tau_n(t), \quad t \in \mathcal{T}, \quad n \in \mathcal{N};
\]

3. **valley-filling,** if there exists \( A \in \mathbb{R} \) such that

\[
\sum_{n \in \mathcal{N}} r_n(t) = [A - D(t)]^+, \quad t \in \mathcal{T}.
\]

**Remark 1.** Optimality of a charging profile \( r \) is independent of the choice of the utility function \( U \) (proved in Theorem 4). That is, if \( r \) is optimal with respect to a strictly convex utility function, then it is optimal with respect to any other strictly convex utility function. Therefore, we can choose \( U(x) = x^2 \) without loss of generality, and see that optimal charging profiles minimize the \( l_2 \) norm of the total load profile. Since the \( l_1 \) norm is a constant for all feasible \( r \) due to (8), minimizing the \( l_2 \) norm “flattens” the total load profile.

**Remark 2.** If the objective is to track a given load profile \( G \) rather than to flatten the total load, we can change the objective function to

\[
\sum_{t=1}^{T} U\left(D(t) + \sum_{n=1}^{N} r_n(t) - G(t)\right)
\]

without affecting the results [3]. For ease of presentation, we focus on the objective function in (5).

**B. Optimal Charging Profile**

**Theorem 3.** If a feasible charging profile \( r \) is valley-filling, then it is optimal.

Valley-filling is our intuitive notion of optimality. However, it may not be always achievable. For example, the “valley” in inelastic base load may be so deep that even if all EVs charge at their maximum rate, the valley still cannot be completely filled. Besides, EVs may have stringent deadlines such that the potential for shifting the load over time to yield valley-filling is limited. The notion of optimality in Definition 1 takes care of these cases and agrees with the intuitive notion of optimality when valley-filling is achievable.

Since the objective function \( U \) depends on \( r \) only through its aggregate \( R_r \), optimal charging profile are clearly nonunique. Let \( \mathcal{O} \) be the set of all optimal points.

**Theorem 4.** The set \( \mathcal{O} \) of optimal charging profiles does not depend on the choice of \( U \). That is, if \( r^* \) is optimal with respect to a strictly convex utility function, then \( r^* \) is also optimal with respect to any other strictly convex utility function.

We now propose a decentralized algorithm for computing optimal charging profiles as the solution to the optimal control problem OC. By decentralized, we mean that EVs choose their own charging profiles, instead of being instructed by a centralized infrastructure. The utility only uses control signals, e.g., prices, to guide EVs’ decisions. We assume that all EVs are available for negotiation at the beginning of the scheduling horizon (even though they are not necessarily available for charging as reflected by time-varying \( \tau_n \)). Figure 1 shows the information exchange between the utility and the EVs for the implementation of this algorithm. Given the “price” profile broadcast by the utility, each EV chooses its charging profile independently, and reports back to the utility. The utility guides EVs’ decision-making by altering the “price” profile. We assume \( U' \) is Lipschitz with the Lipschitz constant \( \beta > 0 \), i.e.,

\[
|U'(x) - U'(y)| \leq \beta|x - y|
\]

for all \( x, y \).

**Algorithm 3:**

Given scheduling horizon \( \mathcal{T} \), base load profile \( D \), the number
Theorem 5. Charging profiles converge to optimal charging profiles, i.e., $r^k \rightarrow O$ as $k \rightarrow \infty$. Furthermore, optimal charging profiles have the same aggregate charging profile $R^{op_k}$, and aggregate charging profiles converge to it, i.e., $R^k \rightarrow R^{op_k}$ as $k \rightarrow \infty$.

IV. OPTIMAL LOAD CONTROL

A. Problem formulation

Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{C}$ denote the set of complex numbers. A variable without a subscript usually denotes a vector with appropriate components, e.g., $d := (d_i, i \in L(j))$, $\omega := (\omega_j, j \in V)$, $P := (P_{ij}, (i, j) \in \mathcal{E})$. For a matrix $A$, $A^\ast$ denotes its transpose and $A^{**}$ its complex conjugate transposed.

1) Transmission network model: The transmission network is described by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{1, \ldots, N\}$ is the set of buses and $\mathcal{E}$ is the set of transmission lines connecting the buses. We adopt the following assumptions 1

- The lines $(i, j) \in \mathcal{E}$ are lossless and characterized by reactances $x_{ij}$.
- The bus voltage magnitudes $|V_i|$ are constant.
- Reactive power is ignored.

We assume that $\mathcal{E}$ is directed, with an arbitrary orientation, so that $(j, i) \notin \mathcal{E}$ if $(i, j) \in \mathcal{E}$. We use $(i, j)$ and $i \rightarrow j$ interchangeably to denote a link in $\mathcal{E}$. We also assume without loss of generality that $\mathcal{G}$ is connected. To simplify notation, we assume all variables represent deviations from their nominal (operating) values and are in per unit.

The dynamics at bus $i$ with a generator is modeled by the swing equation

$$M_j \dot{\omega}_j = P^m_j - P^e_j$$

where $\omega_j$ is the frequency deviation from its nominal value, $M_j$ is the inertia constant of the generator, $P^m_j$ is the deviation in mechanical power injection to bus $i$ from its nominal value, and $P^e_j$ is the deviation in electric power from its nominal value. Each bus may have two types of loads, frequency-sensitive (e.g. motor-type) loads and frequency-insensitive (but controllable) loads. The total change in frequency-sensitive loads at bus $i$ when the frequency deviation is $\omega_i$ is $d_i := D_i \dot{\omega}_i$ where $D_i$ is the damping constant. Let $\mathcal{L}(j)$ denote the set of frequency-sensitive, controllable loads at bus $j$, and $(d_{ij}, i \in \mathcal{L}(j))$ denote the deviations (from their nominal values) of frequency-insensitive loads on bus $j$. Then the electric power $P^e_j$ is the sum of all frequency-sensitive loads, frequency-insensitive loads, and power flows from bus $i$ to other buses

$$P^e_j = D_j \dot{\omega}_j + \sum_{i \in \mathcal{L}(j)} d_i + \sum_{j \rightarrow k} P_{jk} - \sum_{i \rightarrow j} P_{ij}$$

Here $P_{ij}$ is the deviation (from its nominal value) of branch flow from bus $i$ to bus $j$. Our goal is to control the frequency-insensitive loads $d_i$ in response to disturbances $P^m_i$ in generation power. The swing equation can thus be rewritten as

$$\dot{\omega}_j = -\frac{1}{M_j} \left( \sum_{i \in \mathcal{L}(j)} d_i + D_j \omega_j - P^m_j - P^{out}_j + P^{in}_j \right),$$

1This is similar to the standard DC approximation except that we do not assume the phase angle difference is small across each link.
where \( P_{j}^{\text{out}} := \sum_{j \rightarrow k} P_{jk} \) and \( P_{j}^{\text{in}} := \sum_{i \rightarrow j} P_{ij} \) are total branch power flows out and into bus \( j \), respectively.

We assume that the branch flows follow the dynamics
\[
\dot{P}_{ij} = B_{ij} \omega^0 (\omega_i - \omega_j),
\]
where \( \omega^0 \) is the common nominal frequency on which the per-unit convention is based, and
\[
B_{ij} := \frac{|V_i| |V_j|}{x_{ij}} \cos (\theta_i^0 - \theta_j^0).
\]
The dynamic model (12)–(13) is motivated by the following model of deviations in branch flows \( P_{ij} \) when the deviations are small [8] [9, Chapter 11]:
\[
P_{ij} = B_{ij} (\theta_i - \theta_j)
\]
where \( \theta_i \) are the phase angle deviations of the bus voltages, i.e., the voltage phasors are \( V_i := |V_i| e^{j(\theta_i^0 + \theta_i)} \) with the nominal phase angles \( \theta_i^0 \). While the model (14) assumes that the differences \( \theta_i - \theta_j \) of the deviations are small, it does not assume the differences \( \theta_i^0 - \theta_j^0 \) of their nominal values are small.

In summary, the dynamic model of the transmission network is specified by (11)–(13). In steady state, the mechanical power deviations \( P_m \) are equal to the electric power deviations \( P_e \), so \( \dot{\omega}_i = 0 \) and \( \dot{P}_{tk} = 0 \).

2) Optimal load control: Suppose a step change \( P_m := (P_1^m, \ldots, P_N^m) \) in generation is injected to the \( N \) buses.\(^2\) How should the frequency-insensitive loads \( d := (d_i, i \in \mathcal{L}(i), i = 1, \ldots, N) \) in the network be reduced (or increased) in real time in a way that (i) balances the generation shortfall (or surplus), (ii) resynchronizes the bus frequencies, and (iii) minimizes the aggregate disutility of load control? We now formulate this as an optimal load control (OLC) problem.

The disturbance \( P_m \) in generation causes a nonzero frequency deviation \( \omega_i \). This incurs a cost to frequency-sensitive loads and suppose this cost is \( \frac{1}{2D_j} d_i^2 \) in total at bus \( i \). Suppose the frequency-insensitive load \( \ell \) is to be changed by an amount \( d_\ell \) and this will incur a cost (disutility) of \( c_\ell (d_\ell) \). We assume \( -\infty < d_\ell < \bar{d}_\ell < \infty \). Our goal is to minimize the total cost over \((d, d')\) while balancing generation and load across the network:

**OLC**
\[
\begin{aligned}
\min_{d \leq d' \leq \bar{d}} & \quad \sum_{j=1}^{N} \left( \sum_{\ell \in \mathcal{L}(j)} c_\ell (d_\ell) + \frac{1}{2D_j} d_j^2 \right) \\
\text{subject to} & \quad \sum_{j=1}^{N} \left( \sum_{\ell \in \mathcal{L}(j)} d_\ell + d_j \right) = \sum_{j=1}^{N} P_{j}^{m}
\end{aligned}
\]

**Remark 3.** Note that (16) does not require balance of generation and load at each individual bus, but only balance across the entire network. This is less restrictive and offers more opportunity to minimize costs. Additional constraints can be imposed if it is desirable that certain buses balance their own supply and demand, e.g., for economic or regulatory reasons.

We make the following assumptions:

C0: OLC is feasible.
C1: The cost functions \( c_\ell \) are strictly convex and twice continuously differentiable on \([d_\ell, \bar{d}_\ell] \).

B. Load control and swing dynamics as primal-dual solution

The objective function of the dual problem of OLC is
\[
\Phi_J(\nu) := \sum_{j=1}^{N} \min_{\nu \in E} \left( \sum_{\ell \in \mathcal{L}(j)} c_\ell (d_\ell) - \nu d_\ell \right) + \frac{1}{2D_j} d_j^2 - \nu d_j + \nu P_{j}^{m}.
\]

Hence,
\[
\Phi_J(\nu) := \sum_{\ell \in \mathcal{L}(j)} c_\ell (d_\ell (\nu)) - \nu d_\ell (\nu) - \frac{1}{2D_j} d_j^2 - \nu d_j + \nu P_{j}^{m},
\]
where
\[
d_\ell (\nu) := \left[ c_\ell^{-1} (\nu) \right] \bar{d}_\ell.
\]

This objective function has a scalar variable \( \nu \) and is not separable across buses \( j \). Its direct solution hence requires coordination along all buses. A distributed version of the dual problem where each bus \( j \) optimizes its own variable \( \nu_j \) that are constrained to be equal at optimality is the following:

**DOLC**
\[
\max_{\nu_j} \quad \Phi(\nu) := \sum_{j=1}^{N} \Phi_J(\nu_j)
\]

subject to
\[
\nu = \nu_j \quad \text{for all } (i, j) \in \mathcal{E}.
\]

**Theorem 6.**
1) **DOLC has a unique optimal solution** \( \nu^* \) with \( \nu_i^* = \nu_j^* = \nu^* \).\(^3\)
2) **OLC has a unique optimal solution** \((d^*, \bar{d}^*)\) where \( d_i^* = d_i^* (\nu^*) \) is given by (18) and \( d_i^* = D_i \nu^* \).
3) **There is no duality gap.**

Instead of solving OLC directly, Theorem 6 suggests solving its dual DOLC and recovering the unique optimal solution \((d^*, \bar{d}^*)\) of the primal problem OLC from the unique dual optimal \( \nu^* \). To derive a distributed solution for DOLC, consider its Lagrangian
\[
L(\nu, \pi) := \sum_{j=1}^{N} \Phi_J(\nu_j) - \sum_{i \rightarrow j} \pi_{ij} (\nu_i - \nu_j)
\]
where \( \nu \) is the (vector) variable for DOLC and \( \pi \) is the associated dual variable for the dual of DOLC. Hence \( \pi_{ij} \), for all \((i, j) \in \mathcal{E} \), measure the cost of not synchronizing

\(^2\)If there is no generator at bus \( i \), then \( P_{i}^{m} = 0 \).

\(^3\)We abuse notation and use \( \nu^* \) to denote both the vector and the common value of its components.
Theorem 7. Any trajectory \((d(t), \dot{d}(t), \omega(t), P(t))\) converges to a limit \((d^*, \dot{d}^*, \omega^*, P^*)\) such that

1) \((d^*, \dot{d}^*)\) is the unique vector of optimal load control for OLC;
2) \(\omega^*\) is the unique vector of optimal frequency deviations for DOLC;
3) \(P^*\) is a vector of optimal branch flows for the dual of DOLC.

REFERENCES