Performance Improvement of Closed-Form Joint Diagonalizer of Non-Negative Hermitian Matrices

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Abstract—**Joint diagonalization of a series of non-negative Hermitian matrices is one of important techniques in the fields of signal processing, such as blind source separation based on second order statistics. In our previous works, we introduced a closed-form solution of a joint diagonalizer of non-negative Hermitian matrices and also proposed a method for improving the performance of the solution for the cases where given series of Hermitian matrices are not jointly diagonalizable strictly. However, the performance of the method may degrade when the number of given Hermitian matrices are comparatively small. In this paper, we propose an improved version of the closed-form joint diagonalizer of given set of Hermitian matrices by increasing the number of Hermitian matrices virtually. Some numerical examples are also shown to verify the efficacy of the proposed method.**

I. Introduction

Joint diagonalization of a series of non-negative Hermitian matrices is one of important techniques in the fields of signal processing, such as blind source separation problems based on second order statistics (see [1] for instance). The theories of joint diagonalization of given series of Hermitian matrices are thoroughly investigated in [2]. On the basis of these theories, we gave a closed-form solution for the second-order-statistics-based blind source separation, that is, joint diagonalizer of given series of correlation matrices obtained from observations; and gave a necessary and sufficient condition for the closed-form joint diagonalizer to achieve the source separation in [3]. Since computational costs of the closed-form solution is quite small, it was adopted as an initial value for higher-order-statistics-based blind source separation algorithms (see [4] for instance).

In practical problems, given series of Hermitian matrices are not always jointly diagonalizable strictly. For instance in blind source separation problems, although source signals are assumed to be independent each other, the source signals may have small correlations in practice. In order to deal with this problem, we analyzed perturbations of the solutions caused by these correlations and proposed an improved version of the solution on the basis of the analyses [5]. However, the performance of the improved version of the solution may degrade when the number of given Hermitian matrices are comparatively small.

This work was partially supported by the Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Scientific Research (C), 24500001.

In this paper, we propose an improved version of the method given in [5]. The key idea of the proposed method is to increase the number of Hermitian matrices virtually by considering linear combinations of given matrices. Some numerical examples are also shown to verify the efficacy of the proposed method.

II. Problem Formulation of Joint Diagonalization and Closed-Form Joint Diagonalizer

Let $R_k \in \mathbf{C}^{n\times n},\ (k \in \{1,\dots,K\})$ be a given set of non-negative (*n.n.d.*) Hermitian matrices modeled as follows:

$$R_k = A\Lambda_k A^*, \tag{1}$$

where $A \in \mathbf{C}^{n\times m}$ denotes some full column rank constant matrix, which implies that $n \geq m$, $\Lambda_k \in \mathbf{R}^{m\times m}$ denotes a positive definite (*p.d.*) diagonal matrix, and the superscript * denotes the adjoint (conjugate and transposition) operator, respectively. The aim of joint diagonalization is to obtain a full row rank matrix $W \in \mathbf{C}^{m\times n}$ that makes WR_kW^* diagonal for all $k \in \{1,\dots,K\}$.

Here, we introduce fundamental theorems for the joint diagonalization shown in [2].

Theorem 1: [2] Let $M_1, M_2 \in \mathbf{C}^{n\times n}$ be Hermitian matrices. There exists a unitary matrix $U \in \mathbf{C}^{n\times n}$ that makes U^*M_1U and U^*M_2U diagonal matrices, if and only if

$$M_1M_2 = M_2M_1 \tag{2}$$

holds.

Theorem 2: [2] Let $R_k \in \mathbf{C}^{n\times n},\ (k \in \{1,\dots,K\})$ be *n.n.d.* Hermitian matrices and let

$$B = \sum_{k=1}^{K} R_k.$$

There exists a non-singular matrix T that makes T^*R_kT diagonal for all $k \in \{1,\dots,K\}$, if and only if

$$R_iB^-R_j = R_jB^-R_i \tag{3}$$

holds for any $i,j \in \{1,\dots,K\}$, where B^- denotes an arbitrary generalized inverse matrix of B [2], that is, a matrix B^- satisfying $BB^-B = B$.

On the basis of these theorems, we gave a closed-form solution of W in [3]. Here, we give an overview of the solution. Let $\tilde{A} \in \mathbf{C}^{n\times(n-m)}$ be a full column rank matrix consisting of basis vectors of $\mathcal{N}(A^*)$ (null space of A^*). Then

$$\hat{A} = \left[\begin{array}{cc} A & \tilde{A} \end{array}\right] \in \mathbf{C}^{n\times n}$$

is reduced to a non-singular matrix. Also let

$$\hat{\Lambda}_k = \left[\begin{array}{cc} \Lambda_k & O_{m,n-m} \\ O_{n-m,m} & O_{n-m,n-m} \end{array}\right] \in \mathbf{R}^{n\times n},$$

where $O_{m,n}$ denote the zero matrix in $\mathbf{C}^{m\times n}$. Then (1) can be represented as

$$R_k = \hat{A}\hat{\Lambda}_k\hat{A}^*. \tag{4}$$

Although $\hat{A}$ is unknown, its inverse surely exists and it jointly diagonalizes all R_k. Therefore from Theorem 2,

$$R_iB^-R_j = R_jB^-R_i \tag{5}$$

holds for any $i, j \in \{1, \ldots, K\}$, where $B = \sum_{k=1}^{K} R_k$. Note that since $R_iB^-R_j$ is invariant for any B^- [2], we adopt B^+ for B^-, where B^+ denotes the Moore-Penrose generalized inverse matrix of B [2]. Let $B^+ = LL^*$ be a full-rank decomposition of B^+ with $L \in \mathbf{C}^{n\times m}$. Then (5) is rewritten as

$$R_iLL^*R_j = R_jLL^*R_i \tag{6}$$

and we have

$$(L^*R_iL)(L^*R_jL) = (L^*R_jL)(L^*R_iL), \tag{7}$$

which implies that L^*R_iL and L^*R_jL are commutable for any $i, j \in \{1, \ldots, K\}$. Thus, it is concluded that all $R_k,\ (k \in \{1, \ldots, K\})$ can be diagonalized by the same unitary matrix from Theorem 1. Let

$$L^*R_kL = UD_kU^* \tag{8}$$

be the eigenvalue decomposition of R_k. Note that the unitary matrix U does not depend on k from the above discussion. Thus,

$$W = (LU)^* \tag{9}$$

gives a closed-form solution of a joint diagonalizer of $R_k,\ (k \in \{1, \ldots, K\})$. Note that the computational order of these procedures is $O(n^3 + m^3)$ since it is dominated by the calculation of B^+ and that of eigenvalue decomposition of L^*R_kL for a certain $k \in \{1, \ldots, K\}$. When we consider the blind source separation based on second order statistics, calculation of correlation matrices need $O(Tn^2)$, where T denotes the length of observations. In general, $T \gg n$ holds. Therefore, computational costs for obtaining the closed-form joint diagonalizer W is much less than those for obtaining correlation matrices. Due to its small computational costs, this closed-form solution was adopted as an initial value for blind source separation methods based on higher order statistics in [4].

III. Improved Version of Joint Diagonalizer

In practical problems, a given set of Hermitian matrices are not always jointly diagonalizable strictly. For instance in blind source separation problems, although unknown source signals are assumed to be independent, they may have small correlations. In [5], we analyzed the perturbation of the closed-form solution, given in the previous section, caused by these small correlations and proposed a method for improving the performance of the solution. In this section, we briefly review this improved method.

We modeled a given set of Hermitian matrices that can not be jointly diagonalizable strictly as follows:

$$\tilde{R}_k = A\Lambda_k^{1/2}(I_m + C_k)\Lambda_k^{1/2}A^*,\ (k \in \{1, \ldots, K\}), \tag{10}$$

where I_m denotes the identity matrix of degree n, and $C_k = (c_{ij}^{(k)})$ denotes a Hermitian matrix whose diagonal elements are zeros and non-diagonal elements are *i.i.d* complex random variables with zero-mean and variance $E|c_{ij}^{(k)}|^2 = \sigma^2$. Accordingly, $\tilde{R}_k$ can be regarded as a perturbed version of R_k by

$$Z_k = A\Lambda_k^{1/2}C_k\Lambda_k^{1/2}A^*.$$

Since $c_{ij}^{(k)}$ is assumed to be *i.i.d.* and zero mean while Λ_k is non-negative, the perturbation in $\tilde{B} = \sum_{k=1}^{K} \tilde{R}_k$ is relatively small. Therefore, we assume that $\tilde{B} = \sum_{k=1}^{K} \tilde{R}_k \simeq B$ and $\tilde{B}^+ \simeq B^+ = LL^*$. Due to the existence of C_k in (10), (8) does not hold for $\tilde{R}_k$, which implies that the unitary matrix in the eigenvalue decomposition of $L^*\tilde{R}_kL$ depends on k, that is,

$$L^*\tilde{R}_kL = \tilde{U}_k\tilde{D}_k\tilde{U}_k^*. \tag{11}$$

Accordingly, we have K candidates of joint diagonalizer of $\tilde{R}_k,\ (k \in \{1, \ldots, K\})$), written as $W_k = (L\tilde{U}_k)^*$ and we have to select one among them.

If Z_k is comparatively small, the eigenvalue decomposition of $L^*\tilde{R}_kL$ can be represented as

$$L^*\tilde{R}_kL = (U + dU_k)(D_k + dD_k)(U + dU_k)^*, \tag{12}$$

where $U + dU_k$ and $D_k + dD_k$ denote the first-order Taylor expansions of $\tilde{U}_k$ and $\tilde{D}_k$. Therefore, if we select $\tilde{U}_k$ with the smallest $||dU_k||_F^2$, then it is expected to the optimal one, where $||\cdot||_F$ denotes the Frobenius norm of a matrix. However, since dU_k is unknown, we can not obtain $||dU_k||_F^2$. In [5], we had the formula of the expectation of $||dU_k||_F^2$ as follows:

Theorem 3: [5]

$$E_{C_k}||dU_k||_F^2 = \sigma^2 \sum_{i=1}^{m} \sum_{j=1, j\neq i}^{m} \frac{\lambda_i^{(k)}\lambda_j^{(k)}}{(\lambda_i^{(k)} - \lambda_j^{(k)})^2}, \tag{13}$$

where $\lambda_i^{(k)}$ denotes the i-th eigenvalue of L^*R_kL.

Since σ^2 does not depend on k, the optimal selection is given as

$$k_{opt} = \arg\min_k J(k), \tag{14}$$

where

$$J(k) = \sum_{i=1}^{m} \sum_{j=1, j\neq i}^{m} \frac{\lambda_i^{(k)} \lambda_j^{(k)}}{(\lambda_i^{(k)} - \lambda_j^{(k)})^2}. \quad (15)$$

Finally, we have a joint diagonalizer, that is expected to be optimal in terms of the expectation over C_k, as

$$W_{opt} = (L\tilde{U}_{k_{opt}})^*. \quad (16)$$

Note that since D_k is unknown, we can not calculate $J(k)$ directly. Here, we introduce the following theorem in order to analyze the properties of dD_k.

Theorem 4: [6] Let X_0 be an $n \times n$ Hermitian matrix. Let $\boldsymbol{u}_0$ be a normalized eigenvector associated with a simple eigenvalue λ_0 of X_0. Then a complex-valued function λ and a vector-valued function $\boldsymbol{u}$ are defined for all X in some neighborhood $B(X_0) \in \mathbf{C}^{n\times n}$ of X_0, such that

$$\lambda(X_0) = \lambda_0,\ \boldsymbol{u}(X_0) = \boldsymbol{u}_0,$$
$$X\boldsymbol{u} = \lambda \boldsymbol{u},\ \boldsymbol{u}_0^* \boldsymbol{u} = 1\ (X \in B(X_0)).$$

Moreover, the functions λ and $\boldsymbol{u}$ are ∞ times differentiable on $B(X_0)$, and the differentials at X_0 are

$$d\lambda = \boldsymbol{u}_0^* (dX) \boldsymbol{u}_0, \quad (17)$$

and

$$d\boldsymbol{u} = (\lambda_0 I_n - X_0)^+ (dX) \boldsymbol{u}_0. \quad (18)$$

According to this theorem, the first-order Taylor expansion of i-th eigenvalue of $\tilde{D}_k$ can be written as

$$\tilde{\lambda}_i^{(k)} = \lambda_i^{(k)} + (\boldsymbol{u}_i^{(k)})^* L^* Z_k L \boldsymbol{u}_i^{(k)}, \quad (19)$$

where $\boldsymbol{u}_i^{(k)}$ denotes the i-th column of U, that is, the eigen vector of $L^* R_k L$ corresponding to the eigenvalue $\lambda_i^{(k)}$. Since C_k is assumed to be zero-mean, the expectation of $\tilde{\lambda}_i^{(k)}$ is also reduced to zero. Therefore, we use $\tilde{\lambda}_i^{(k)}$ instead of $\lambda_i^{(k)}$ to calculate $J(k)$.

In [5], it was confirmed that this selection scheme works well especially in case that K is comparatively large. On the other hand, when K is comparatively small, the performance of this method may degrade.

Note that the computational order of this improved version is $O(n^3 + Km^3)$ since we have to calculate the eigenvalue decompositions for all $L^* \tilde{R}_k L,\ (k \in \{1, \ldots, K\})$.

IV. The Proposed Method

In this section, we construct an improved version of the method proposed in [5], in which the number of candidates is virtually increased by considering linear combinations of given Hermitian matrices.

Let $\mathcal{K}_0 = 2^{\{1,\ldots,K\}}$ be the power set of $\{1, \ldots, K\}$ and let $\mathcal{K} = \mathcal{K}_0 - \{\phi, \{1, \ldots, K\}\}$. Note that

$$|\mathcal{K}| = 2^K - 2$$

holds. Therefore, we can construct at most $2^K - 2$ Hermitian matrices written as

$$\tilde{R}_S = \sum_{k\in S} \tilde{R}_k,\ S \in \mathcal{K} \quad (20)$$

that is a linear combination of given Hermitian matrices. Hereafter, we also use $S \in \mathcal{K}$ as indices as the same with $k \in \{1, \ldots, K\}$. Also we have at most $2^K - 2$ candidates for $\tilde{U}_S$ obtained by the eigenvalue decomposition

$$L^* \tilde{R}_S L = \tilde{U}_S \tilde{D}_S \tilde{U}_S^*,\ S \in \mathcal{K}. \quad (21)$$

However, some candidates $\tilde{U}_S,\ (S \in \mathcal{K})$ may not be independent. In fact, the following theorem holds.

Theorem 5: Let $S \in \mathcal{K}$ and T be the complement set of S in $\{1, \ldots, K\}$, then

$$(L^* \tilde{R}_S L)(L^* \tilde{R}_T L) = (L^* \tilde{R}_T L)(L^* \tilde{R}_S L) \quad (22)$$

holds.

Proof: Note that $BB^+ = B^+B$ holds since $B = \sum_{k=1}^{K} \tilde{R}_k$ is Hermitian, and it is trivial that $BB^+ = B^+B$ is the orthogonal projector onto the range space of B, written as $\mathcal{R}(B)$. Also note that $BB^+\tilde{R}_S = \tilde{R}_S BB^+ = \tilde{R}_S$ holds for any $S \in \mathcal{K}$ since $\mathcal{R}(\tilde{R}_S) \subset \mathcal{R}(B)$ for any $S \in \mathcal{K}$.

Since $\tilde{R}_T = B - \tilde{R}_S$ and $B = LL^*$, we have

$$\begin{aligned}
&(L^*\tilde{R}_S L)(L^*\tilde{R}_T L) \\
&= (L^*\tilde{R}_S L)(L^*(B - \tilde{R}_S)L) \\
&= L^*\tilde{R}_S LL^*BL - L^*\tilde{R}_S LL^*\tilde{R}_S L \\
&= L^*\tilde{R}_S B^+BL - L^*\tilde{R}_S B^+\tilde{R}_S L \\
&= L^*\tilde{R}_S L - L^*\tilde{R}_S B^+\tilde{R}_S L.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&(L^*\tilde{R}_T L)(L^*\tilde{R}_S L) \\
&= L^*\tilde{R}_S L - L^*\tilde{R}_S B^+\tilde{R}_S L,
\end{aligned}$$

which concludes the proof. ■

According to this theorem and Theorem 1, it is concluded that $L^*\tilde{R}_S L$ and $L^*\tilde{R}_T L$ share all eigen vectors, that is, $\tilde{U}_S = \tilde{U}_T$. Therefore, we have $2^{K-1} - 1$ independent candidates of $\tilde{U}_S$.

The rest problem that should be resolved is which of $J(S)$ and $J(T)$ should be adopted to evaluate the goodness of $\tilde{U}_S = \tilde{U}_T$.

Since $B = \sum_{k=1}^{K} R_k$, we have

$$L^*BL = I_m = \sum_{k=1}^{K} L^* R_k L = U \left(\sum_{k=1}^{K} D_k \right) U^*, \quad (23)$$

which implies that

$$I_m = \sum_{k=1}^{K} D_k,\ \sum_{k=1}^{K} \lambda_i^{(k)} = 1. \quad (24)$$

Also we have,

$$I_m = D_S + D_T,\ \lambda_i^{(S)} + \lambda_i^{(T)} = 1 \quad (25)$$

for $S \in \mathcal{K}$ and its complement $T \in \mathcal{K}$.

The criterion (15) for $S \in \mathcal{K}$ is written as

$$J(S) = \sum_{i=1}^{m} \sum_{j=1, j\neq i}^{m} \frac{\lambda_i^{(S)} \lambda_j^{(S)}}{(\lambda_i^{(S)} - \lambda_j^{(S)})^2} \quad (26)$$

and that for T is written as

$$J(T) = \sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m}\frac{(1-\lambda_i^{(S)})(1-\lambda_j^{(S)})}{((1-\lambda_i^{(S)})-(1-\lambda_j^{(S)}))^2} = J(S)+\sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m}\frac{1-\lambda_i^{(S)}-\lambda_j^{(S)}}{(\lambda_i^{(S)}-\lambda_j^{(S)})^2}. \quad (27)$$

Since we have no *a priori* information for $\lambda_i^{(k)}$, it is natural to assume that $\lambda_i^{(k)}$, $(k \in \{1,\ldots,K\}, i \in \{1,\ldots,m\})$ are $i.i.d.$ random variables with the constraint $\lambda_i^{(k)} \geq 0$. Therefore, we have $E[\lambda_i^{(k)}] = 1/K$ for all $i \in \{1,\ldots,m\}$ and $E[\lambda_i^{(S)}] = |S|/K$ and $E[\lambda_i^{(T)}] = 1-|S|/K$. Accordingly,

$$E[1-\lambda_i^{(S)}-\lambda_i^{(S)}] = 1-2|S|/K \quad (28)$$

is obtained which implies that $J(S)$ tends to be smaller than $J(T)$ when $|S| < |T|$ (which means $|S| < |\mathcal{K}|/2$). Thus, it is concluded that we had better to adopt $J(S)$ to evaluate the goodness of $\tilde{U}_S = \tilde{U}_T$ when $|S| < |T|$.

Note that the set of Hermitian matrices obtained by linear combination includes R_k, $(k = \{1,\ldots,K\})$ (linear combination of one matrix). Therefore, if we can select the best matrix*, the proposed method always outperforms the conventional one proposed in [5].

Also note that the approximated computational order of this improved version is $O(n^3 + 2^{K-1}m^3)$ when we adopt all independent candidates $\tilde{R}_S \in \mathcal{K}$. Note that we can intensionally reduce the number of candidates when K is comparatively large in order to reduce the computational costs. In such cases, it is suggested to adopt linear combinations with small number of Hermitian matrices on the basis of the above analyses for $J(S)$.

V. Computer Simulations

In this section, we verify the efficacy of the proposed method by computer simulations. We compare the performance of the proposed joint diagonalizer and our previous method proposed in [5]. Let W_c and W_p denotes the joint diagonalizer obtained by the conventional method [5] and that obtained by the proposed method. As the evaluation measure, we adopt

$$Z(W) = \frac{||(WA)(WA)^* - \mathrm{diag}((WA)(WA)^*)||_F^2}{||(WA)(WA)^*||_F^2}, \quad (29)$$

where $\mathrm{diag}(X)$ denotes the diagonal matrix whose diagonal elements are the same with X. Note that $Z(W) \geq 0$ holds and a smaller $Z(W)$ means a better result. Also note that $Z(W) = 0$ is achieved when $W = PDA_\ell^{-1}$, where P, D, and A_ℓ^{-1} denote an arbitrary permutation matrix, an arbitrary full-rank diagonal matrix, and an arbitrary left inverse matrix of A, respectively, which implies that W achieves perfect joint diagonalization. We randomly generate A, Λ_k, and C_k with the conditions $\sigma = 0.01$, $m = n = 4, 8$, and comparatively small $K = 5, 9$. In the proposed method, we have 15 candidates for $K = 5$ (all possible candidates) and $45 = {}_9C_1 + {}_9C_2$ candidates for $K = 9$ including linear combinations of two Hermitian matrices at most.

*Selection of the best matrix may not be achieved by using $J(S)$ since $J(S)$ is an approximation of $E_{C_S}||dU_S||_F^2$.

TABLE I
Means and variances of A_c.

$K=5$	mean of A_c	variance of A_c
$m=n=4$	1.39×10^{-2}	1.35×10^{-1}
$m=n=8$	3.50×10^{-2}	2.12×10^{-1}

$K=9$	mean of A_c	variance of A_c
$m=n=4$	2.99×10^{-2}	2.08×10^{-1}
$m=n=8$	5.25×10^{-2}	2.57×10^{-1}

Table I shows the mean and the variance of

$$A_c = \frac{Z(W_c) - Z(W_p)}{Z(W_c)} \quad (30)$$

over $1,000$ trials for each (m, K). Note that a positive A_c means that the proposed method outperforms the conventional one. According to Table I, it is confirmed that the proposed method outperforms the conventional one in terms of the mean since the mean of A_c is positive. On the other hand, it is confirmed that the variance of A_c is comparatively large, which implies that the selection criterion $J(S)$ may fail to select the best one. Thus, the improvement of the selection criterion $J(S)$ is needed in order to improve the overall algorithm.

VI. Conclusion

In this paper, we proposed an improved version of the closed-form joint diagonalizer of a given set of *n.n.d* Hermitian matrices by incorporating linear combinations of given Hermitian matrices. We also investigated the performance of the proposed method by computer simulations and confirmed the efficacy of the proposed method. Improvement of the selection criterion is one of our future works that should be undertaken.

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