Design of FIR Fractional Delay Filter Based on Maximum Signal-to-Noise Ratio Criterion

Chien-Cheng Tseng* and Su-Ling Lee†
*Department of Computer and Communication Engineering, National Kaohsiung First University of Science and Technology, Kaohsiung, Taiwan, R.O.C.
E-mail: tcc@nkfust.edu.tw Tel: +886-7-6011000 ext 2029
†Department of Computer Science and Information Engineering, Chang-Jung Christian University, Tainan, Taiwan, R.O.C.
E-mail: lilee@mail.cjcu.edu.tw Tel: +886-6-2785123 ext 6155

Abstract— **In this paper, a new approach to the design of digital FIR fractional delay filter with consideration of noise attenuation is presented. The design is based on the maximization of signal-to-noise ratio (SNR) at the output of the fractional delay filter under the constraint that actual frequency response and ideal response have several same derivatives at the prescribed frequency point. The optimal filter coefficients are obtained from the generalized eigenvector associate with maximum eigenvalue of a pair of matrices. Numerical examples are demonstrated to show the proposed method provides higher SNR than the conventional FIR fractional delay filter design methods without considering the noise attenuation.**

I. Introduction

In recent years, fractional order signal processing has received great attentions in many engineering applications. The research topics include fractional Fourier transform, fractional stochastic processes, fractional calculus and fractional delay filter [1]-[4]. In this paper, the design of fractional delay filter will be studied. So far, the fractional delay filter has become an important element in several applications like beam steering of antenna arrays, time adjustment in digital receivers, modeling of music instruments, speech coding and synthesis, comb filter design, digital differentiator design, digital integrator design, image interpolation and analog to digital conversion, etc. [5]-[12]. An excellent survey of the fractional delay filter design is presented in the tutorial paper [5].

The ideal frequency response of the fractional delay filter is given by

$$H_d(\omega) = e^{-j\omega\tau} \tag{1}$$

where τ is a positive real number in the desired range. Thus, the fractional-delay design problem is how to find a digital filter such that its actual frequency response fits the ideal response $H_d(\omega)$ as well as possible. The existing design techniques of fractional delay filter can be mainly classified into two groups: fixed fractional delay (FFD) design and variable fractional delay (VFD) design. In the former, the delay τ is fixed. The typical FFD design methods include the window method [5], Lagrange interpolation method [5], maximally flat design [7], and discrete Fourier transform method [8]. In the latter, the delay τ is variable. So far, several methods have been proposed to solve the VFD design problem such as Lagrange method [9], weighted least squares (WLS) method [9][10], decoupling approach [11], and second order cone programming [12] etc. These methods have their unique features.

The ideal response in equation (1) is an all-pass filter design specification, so conventional designed FFD and VFD filters are both only suitable for the fractional-sample delay of noise-free measurement signal. When the signal is corrupted by noise, the fractional delay filter needs to be designed with consideration of noise attenuation. One of the methods is to use the cascade method. First, the well-known Wiener filter is used to mitigate the noise. Then, the enhanced signal is fractionally delayed by conventional fractional delay filter. Although, this cascade approach can work well, it needs lager computational load. Moreover, the signal-to-noise ratio (SNR) at the fractional delay filter output is not really maximized. In this paper, the fractional delay filter is designed by maximizing the output SNR under the constraint that actual frequency response and ideal response have several same derivatives at the prescribed frequency point. The optimal filter coefficients are easily obtained from the generalized eigenvector associate with maximum eigenvalue of a pair of matrices.

The organization of this paper is as follows: In section II, the design problem is stated. In section III, the proposed design method is described. In section IV, some numerical examples are demonstrated to show the proposed method provides higher SNR than the conventional fractional delay filter design methods without considering the noise attenuation. Finally, a conclusion remark is made.

II. Design problem

In this paper, the input signal $x(n)$ to the fractional delay filter is a desired signal $s(n)$ corrupted by zero-mean white noise $v(n)$, that is, $x(n) = s(n) + v(n)$. The powers of the signals $s(n)$ and $v(n)$ are assumed to be σ_s^2 and σ_v^2 respectively. The problem considered in this paper is how to design a digital finite impulse response (FIR) fractional delay filter $H(z)$ given by

$$H(z) = \sum_{k=0}^{N} h(k) z^{-k} \quad (2)$$

such that the desired signal $s(n)$ is fractionally delayed and the unwanted noise $v(n)$ is attenuated. Now, let us study the signal to noise ratio (SNR) at the input and output of the fractional delay filter below: If the power spectral densities of the signals $s(n)$ and $v(n)$ are denoted by $S_s(\omega)$ and $S_v(\omega)$, then we have

$$E[s(n)^2] = \tfrac{1}{2\pi}\int_0^{2\pi} S_s(\omega)d\omega = \sigma_s^2 \quad (3a)$$

$$E[v(n)^2] = \tfrac{1}{2\pi}\int_0^{2\pi} S_v(\omega)d\omega = \sigma_v^2 \quad (3b)$$

where operator $E[\cdot]$ is the expected value. Thus, the SNR of the input signal of fractional delay filter is given by

$$SNR_i = \frac{E[s(n)^2]}{E[v(n)^2]} = \frac{\sigma_s^2}{\sigma_v^2} \quad (4)$$

Moreover, the output signal $y(n)$ of the fractional delay filter can be written as

$$y(n) = y_s(n) + y_v(n) \quad (5)$$

where $y_s(n)$ and $y_v(n)$ are the output signals when $s(n)$ and $v(n)$ pass through the fractional delay filter. Thus, the output powers of the signals $y_s(n)$ and $y_v(n)$ are given by

$$E[y_s(n)^2] = \tfrac{1}{2\pi}\int_0^{2\pi} \left|H(e^{j\omega})\right|^2 S_s(\omega)d\omega \quad (6a)$$

$$E[y_v(n)^2] = \tfrac{1}{2\pi}\int_0^{2\pi} \left|H(e^{j\omega})\right|^2 S_v(\omega)d\omega \quad (6b)$$

If the frequency response $H(e^{j\omega})$ approximates the ideal response $H_d(\omega)$ in (1) very well, then we have

$$\left|H(e^{j\omega})\right| \approx \left|H_d(\omega)\right| = \left|e^{-j\omega\tau}\right| = 1 \quad (7)$$

The above expression is valid because the ideal response in equation (1) is an all-pass filter design specification. Substituting (7) into (6), it yields the following results

$$E[y_s(n)^2] \approx \tfrac{1}{2\pi}\int_0^{2\pi} S_s(\omega)d\omega = \sigma_s^2 \quad (8a)$$

$$E[y_v(n)^2] \approx \tfrac{1}{2\pi}\int_0^{2\pi} S_v(\omega)d\omega = \sigma_v^2 \quad (8b)$$

Thus, the SNR of the output signal of fractional delay filter is given by

$$SNR_o = \frac{E[y_s(n)^2]}{E[y_v(n)^2]} = \frac{\sigma_s^2}{\sigma_v^2} \quad (9)$$

Comparing equations (4) and (9), it can be found that the SNR_o is equal to SNR_i for the ideal fractional delay filter. This means that the signal $x(n)$ is fractionally delayed without attenuating the corrupted noise at all. The purpose of this paper is to study how to design FIR fractional delay filter by maximizing the output signal to noise ratio SNR_o if a priori information of the power spectral density function $S_s(\omega)$ is available. The details will be described in next section.

III. PROPOSED DESIGN METHOD

In the proposed method, the filter coefficients $h(k)$ in (2) are determined by maximizing SNR_o at the output of fractional delay filter under the constraint that actual frequency response $H(e^{j\omega})$ and the ideal frequency response $H_d(\omega)$ have several same derivatives at the prescribed frequency point. So, in this section, the computation of SNR_o is first described. Then, the derivative constraints are studied. Finally, the design algorithm is presented.

A. Computation of output SNR

In order to maximize SNR_o, we need to derive a closed-form expression of SNR_o. When the noisy signal $x(n)$ passes through the FIR filter in (2), its output $y(n)$ is

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k) = \boldsymbol{h}^T \boldsymbol{x}(n) \quad (10)$$

where T denotes vector or matrix transpose, vectors $\boldsymbol{h}$ and $\boldsymbol{x}(n)$ are defined by

$$\boldsymbol{h} = [h(0) \quad h(1) \quad \cdots \quad h(N)]^T \quad (11a)$$

$$\boldsymbol{x}(n) = [x(n) \quad x(n-1) \quad \cdots \quad x(n-N)]^T \quad (11b)$$

If we define the desired signal vector $\boldsymbol{s}(n)$ and noise vector $\boldsymbol{v}(n)$ as

$$\boldsymbol{s}(n) = [s(n) \quad s(n-1) \quad \cdots \quad s(n-N)]^T \quad (12a)$$

$$\boldsymbol{v}(n) = [v(n) \quad v(n-1) \quad \cdots \quad v(n-N)]^T \quad (12b)$$

then it can be shown that two components $y_s(n)$ and $y_v(n)$ of the output signal $y(n)$ in (5) can be written as

$$y_s(n) = \boldsymbol{h}^T \boldsymbol{s}(n) \quad (13a)$$

$$y_v(n) = \boldsymbol{h}^T \boldsymbol{v}(n) \quad (13b)$$

Thus, the output powers of $y_s(n)$ and $y_v(n)$ are given by

$$E[y_s(n)^2] = E[(\boldsymbol{h}^T \boldsymbol{s}(n))^2] = \boldsymbol{h}^T \boldsymbol{R}_s \boldsymbol{h} \quad (14a)$$

$$E[y_v(n)^2] = E[(\boldsymbol{h}^T \boldsymbol{v}(n))^2] = \boldsymbol{h}^T \boldsymbol{R}_v \boldsymbol{h} \quad (14b)$$

where two correlation matrices are

$$\boldsymbol{R}_s = E[\boldsymbol{s}(n)\boldsymbol{s}(n)^T] \quad (15a)$$

$$\boldsymbol{R}_v = E[\boldsymbol{v}(n)\boldsymbol{v}(n)^T] \quad (15b)$$

Thus, the output SNR is given by

$$SNR_o = \frac{E[y_s(n)^2]}{E[y_v(n)^2]} = \frac{\boldsymbol{h}^T \boldsymbol{R}_s \boldsymbol{h}}{\boldsymbol{h}^T \boldsymbol{R}_v \boldsymbol{h}} \tag{16}$$

Because $v(n)$ is a white noise, the correlation matrix becomes $\boldsymbol{R}_v = \sigma_v^2 \boldsymbol{I}$, where $\boldsymbol{I}$ is the identity matrix. Thus, the SNR_o in (16) can be written as

$$SNR_o = \frac{\boldsymbol{h}^T \boldsymbol{R}_s \boldsymbol{h}}{\sigma_v^2 \boldsymbol{h}^T \boldsymbol{h}} \tag{17}$$

Usually, the correlation matrix $\boldsymbol{R}_s$ and noise variance σ_v^2 are unknown, so it is necessary to modify the SNR_o in (17) such that the optimization can be performed. The modification is based on the fact: Find parameter vector $\boldsymbol{h}$ to maximize SNR_o is equivalent to find vector $\boldsymbol{h}$ to maximize $\alpha SNR_o + \beta$, where α and β are arbitrary constants and $\alpha > 0$. Thus, choosing $\alpha = \beta = \sigma_v^2$, we have

$$\begin{aligned} J(\boldsymbol{h}) &= \alpha SNR_o + \beta \\ &= \sigma_v^2 \frac{\boldsymbol{h}^T \boldsymbol{R}_s \boldsymbol{h}}{\sigma_v^2 \boldsymbol{h}^T \boldsymbol{h}} + \sigma_v^2 \\ &= \frac{\boldsymbol{h}^T (\boldsymbol{R}_s + \sigma_v^2 \boldsymbol{I}) \boldsymbol{h}}{\boldsymbol{h}^T \boldsymbol{h}} \\ &= \frac{\boldsymbol{h}^T \boldsymbol{R}_x \boldsymbol{h}}{\boldsymbol{h}^T \boldsymbol{h}} \end{aligned} \tag{18}$$

where the $\boldsymbol{R}_x$ is the correlation matrix of the input signal $x(n)$ below:

$$\begin{aligned} \boldsymbol{R}_x &= E[\boldsymbol{x}(n)\boldsymbol{x}(n)^T] \\ &= \boldsymbol{R}_s + \sigma_v^2 \boldsymbol{I} \end{aligned} \tag{19}$$

Because the matrix $\boldsymbol{R}_x$ can be estimated from the input signal $x(n)$, we will find the vector $\boldsymbol{h}$ to maximize $J(\boldsymbol{h})$ in this paper. Note that the optimal parameter $\boldsymbol{h}$ that maximizes $J(\boldsymbol{h})$ also maximizes the signal to noise ratio SNR_o because expression $J(\boldsymbol{h}) = \alpha SNR_o + \beta$ is valid.

B. Computation of derivative constraint

For the fractional delay filter design, the problem is how to find filter coefficient vector $\boldsymbol{h}$ such that the frequency response $H(e^{j\omega})$ approximates the ideal response $H_d(\omega)$ as well as possible. To achieve this purpose, we let the actual frequency response and ideal response have several same derivatives at the prescribed frequency point ω_0. That is, the following derivative constraints are imposed on the frequency response $H(e^{j\omega})$:

$$\left.\frac{d^m H(e^{j\omega})}{d\omega^m}\right|_{\omega=\omega_0} = \left.\frac{d^m H_d(\omega)}{d\omega^m}\right|_{\omega=\omega_0} \tag{20}$$

where the order $m \in \{0,1,2,\cdots,M\}$. So, the total number of constraint is $M+1$. Taking $z = e^{j\omega}$, the frequency response of FIR filter in (2) is given by

$$H(e^{j\omega}) = \sum_{k=0}^{N} h(k) e^{-j\omega k} \tag{21}$$

So, the derivative of actual response can be computed by

$$\begin{aligned} \frac{d^m H(e^{j\omega})}{d\omega^m} &= \frac{d^m \left(\sum_{k=0}^{N} h(k) e^{-j\omega k} \right)}{d\omega^m} \\ &= \sum_{k=0}^{N} h(k) \frac{d^m e^{-j\omega k}}{d\omega^m} \\ &= \sum_{k=0}^{N} h(k)(-jk)^m e^{-j\omega k} \\ &= (-j)^m \sum_{k=0}^{N} h(k) k^m e^{-j\omega k} \\ &= (-j)^m \left(\boldsymbol{h}^T \boldsymbol{c}_R(\omega,m) - j\boldsymbol{h}^T \boldsymbol{c}_I(\omega,m) \right) \end{aligned} \tag{22}$$

where the vectors $\boldsymbol{c}_R(\omega,\upsilon)$ and $\boldsymbol{c}_I(\omega,\upsilon)$ are given by

$$\boldsymbol{c}_R(\omega,m) = \begin{bmatrix} 0^m \\ 1^m \cos(\omega) \\ 2^m \cos(2\omega) \\ \vdots \\ N^m \cos(N\omega) \end{bmatrix} \tag{23a}$$

$$\boldsymbol{c}_I(\omega,m) = \begin{bmatrix} 0 \\ 1^m \sin(\omega) \\ 2^m \sin(2\omega) \\ \vdots \\ N^m \sin(N\omega) \end{bmatrix} \tag{23b}$$

Moreover, using the equation (1), the derivative of the ideal frequency response is computed by

$$\begin{aligned} \frac{d^m H_d(\omega)}{d\omega^m} &= \frac{d^m e^{-j\omega\tau}}{d\omega^m} \\ &= (-j\tau)^m e^{-j\omega\tau} \\ &= (-j)^m \tau^m e^{-j\omega\tau} \\ &= (-j)^m \left(\tau^m \cos(\omega\tau) - j\tau^m \sin(\omega\tau) \right) \end{aligned} \tag{24}$$

Using equations (22) and (24), the constraints in (20) reduce to the following form:

$$\boldsymbol{h}^T \boldsymbol{c}_R(\omega_0, m) = \tau^m \cos(\omega_0 \tau) \tag{25a}$$

$$\boldsymbol{h}^T \boldsymbol{c}_I(\omega_0, m) = \tau^m \sin(\omega_0 \tau) \quad (25b)$$

where $m = 0,1,\cdots,M$. The above constraints can be further written as the matrix form below:

$$\boldsymbol{C}\boldsymbol{h} = \boldsymbol{f} \quad (26)$$

where

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{C}_R \\ \boldsymbol{C}_I \end{bmatrix} \qquad \boldsymbol{f} = \begin{bmatrix} \boldsymbol{f}_R \\ \boldsymbol{f}_I \end{bmatrix} \quad (27)$$

with

$$\boldsymbol{C}_R = \begin{bmatrix} \boldsymbol{c}_R^T(\omega_0,0) \\ \boldsymbol{c}_R^T(\omega_0,1) \\ \vdots \\ \boldsymbol{c}_R^T(\omega_0,M) \end{bmatrix} \qquad \boldsymbol{f}_R = \begin{bmatrix} \tau^0 \cos(\omega_0\tau) \\ \tau^1 \cos(\omega_0\tau) \\ \vdots \\ \tau^M \cos(\omega_0\tau) \end{bmatrix} \quad (28a)$$

$$\boldsymbol{C}_I = \begin{bmatrix} \boldsymbol{c}_I^T(\omega_0,0) \\ \boldsymbol{c}_I^T(\omega_0,1) \\ \vdots \\ \boldsymbol{c}_I^T(\omega_0,M) \end{bmatrix} \qquad \boldsymbol{f}_I = \begin{bmatrix} \tau^0 \sin(\omega_0\tau) \\ \tau^1 \sin(\omega_0\tau) \\ \vdots \\ \tau^M \sin(\omega_0\tau) \end{bmatrix} \quad (28b)$$

Finally, two remarks are made below: First, the equation (25b) is always valid if the $\omega_0 = 0$. Thus, the equation (27) is used for the case of $\omega_0 \neq 0$. If $\omega_0 = 0$ is chosen, equation (27) reduces to $\boldsymbol{C} = \boldsymbol{C}_R$ and $\boldsymbol{f} = \boldsymbol{f}_R$. Second, if $\omega_0 = 0$ and $N = M$ are chosen, the above design reduces to the conventional maximally flat Lagrange design. In this case, the filter coefficients have the closed-form solution below:

$$h(k) = \prod_{\ell=0,\ell\neq k}^{N} \frac{\tau - \ell}{k - \ell} \quad (29)$$

So far, the derivative constraints have been described. In what follows, we will combine the SNR_o maximization and these derivative constraints to design FIR fractional delay filter.

C. Design Algorithm

In this paper, the fractional delay filter $H(z)$ is designed by maximizing the cost function $J(\boldsymbol{h})$ in (18) subject to the derivative constraint $\boldsymbol{C}\boldsymbol{h} = \boldsymbol{f}$ in (26). That is, we want to solve the following optimization problem:

$$\text{Maximize } J(\boldsymbol{h}) = \frac{\boldsymbol{h}^T \boldsymbol{R}_x \boldsymbol{h}}{\boldsymbol{h}^T \boldsymbol{h}}$$

$$\text{Subject to } \boldsymbol{C}\boldsymbol{h} = \boldsymbol{f} \quad (30)$$

This is a constrained nonlinear optimization problem, so it is not easy to obtain the optimum solution. However, it is fortunate that the problem in (30) is approximately equivalent to the following problem:

$$\text{Maximize } \frac{\boldsymbol{h}^T \boldsymbol{R}_x \boldsymbol{h} + \eta}{\boldsymbol{h}^T \boldsymbol{h} + \eta}$$

$$\text{Subject to } \boldsymbol{C}\boldsymbol{h} = \boldsymbol{f} \quad (31)$$

where η is a very small number. The smaller η, the better the approximation. In order to solve the problem in (31), let us define three matrices and a vector as

$$\boldsymbol{Q}_1 = \begin{bmatrix} \boldsymbol{R}_x & \boldsymbol{0} \\ \boldsymbol{0} & \eta \end{bmatrix} \quad \boldsymbol{Q}_2 = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \eta \end{bmatrix}$$

$$\boldsymbol{E} = [\boldsymbol{C} \quad \boldsymbol{f}] \qquad \boldsymbol{a} = \begin{bmatrix} \boldsymbol{h} \\ -1 \end{bmatrix} \quad (32)$$

then the problem in (31) can be rewritten as the form:

$$\text{Maximize } \frac{\boldsymbol{a}^T \boldsymbol{Q}_1 \boldsymbol{a}}{\boldsymbol{a}^T \boldsymbol{Q}_2 \boldsymbol{a}}$$

$$\text{Subject to } \boldsymbol{E}\boldsymbol{a} = \boldsymbol{0} \quad (33)$$

where $\boldsymbol{0}$ is zero matrix. The key idea to solve this constrained optimization problem is that all the vectors $\boldsymbol{a}$ satisfying the constraint $\boldsymbol{E}\boldsymbol{a} = \boldsymbol{0}$ can be expressed as $\boldsymbol{a} = \boldsymbol{B}\boldsymbol{w}$, where the columns of matrix $\boldsymbol{B}$ form an orthonormal basis of the null space of matrix $\boldsymbol{E}$. Using the QR decomposition or SVD, it is easy to get $\boldsymbol{B}$ from $\boldsymbol{E}$. The details can be found in [13]. Thus, the problem in (33) can be further simplified as

$$\text{Maximize } \frac{\boldsymbol{w}^T \boldsymbol{B}^T \boldsymbol{Q}_1 \boldsymbol{B}\boldsymbol{w}}{\boldsymbol{w}^T \boldsymbol{B}^T \boldsymbol{Q}_2 \boldsymbol{B}\boldsymbol{w}} \quad (34)$$

It is well known that the optimal solution $\boldsymbol{w}_o$ of the optimization problem in (34) is the generalized eigenvector of a pair of matrices $\boldsymbol{B}^T\boldsymbol{Q}_1\boldsymbol{B}$ and $\boldsymbol{B}^T\boldsymbol{Q}_2\boldsymbol{B}$ corresponding to the maximum eigenvalue. And, the desired solution $\boldsymbol{a}_o$ is equal to $\boldsymbol{B}\boldsymbol{w}_o$. So, by scaling the vector $\boldsymbol{a}_o$ into the form $\boldsymbol{a}_o = [\boldsymbol{h}_o \quad -1]$, then the vector $\boldsymbol{h}_o$ is an approximate optimum solution of the problem in (30). The smaller η is chosen, the better approximation to the optimum solution has. Now, the entire algorithm to design fractional delay filter is summarized as follows: Given the noisy measurement signal $x(n)$ $(n = 1,2,\cdots,L)$, filter order N, fractional delay τ, number of derivative constraints M, frequency point ω_0, and small number η.

Step 1: Compute matrix $\boldsymbol{C}$ and vector $\boldsymbol{f}$ by using (27).

Step 2: Estimate the correlation matrix $\boldsymbol{R}_x$ by using the time average formula:

$$\boldsymbol{R}_x = \frac{1}{L-N} \sum_{n=N+1}^{L} \boldsymbol{x}(n)\boldsymbol{x}(n)^T \quad (35)$$

where $\boldsymbol{x}(n)$ is defined in (11b).

Step 3: Construct three matrices $\boldsymbol{Q}_1$, $\boldsymbol{Q}_2$, and $\boldsymbol{E}$ by using equation (32).

Step 4: Use the QR decomposition or SVD to find the null space of matrix $\boldsymbol{E}$ and construct matrix $\boldsymbol{B}$.

Step 5: Find the generalized eigenvector of a pair of matrices $\boldsymbol{B}^T\boldsymbol{Q}_1\boldsymbol{B}$ and $\boldsymbol{B}^T\boldsymbol{Q}_2\boldsymbol{B}$ corresponding to the maximum eigenvalue. And, compute the vector $\boldsymbol{a}_o$ using the equation $\boldsymbol{B}\boldsymbol{w}_o$.

Step 6: Scale the vector $\boldsymbol{a}_o$ into the form $\boldsymbol{a}_o = [\boldsymbol{h}_o \quad -1]$, then the vector $\boldsymbol{h}_o$ is an approximate optimum solution.

Step 7: Let the measurement $x(n)$ pass through the designed FIR filter $H(z)$, then its output

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k) \tag{36}$$

is the desired fractional-delay signal of $x(n)$.

Three remarks are made below: First, the proposed method uses the correlation matrix $\boldsymbol{R}_x$ of the input signal $x(n)$ to design fractional delay filter, so it is signal-dependent design. That is, the designed fractional delay filter has considered the characteristic of the signal to attenuate the noise. Second, the number of derivative constraints $M+1$ needs to be smaller than the filter length $N+1$ such that there are $N-M$ degree of freedom to maximize the output SNR. Third, if $s(n)$ is a narrowband signal whose energy concentrates on the frequency band around the point ω_c, then the ω_0 had better be chosen as ω_c to let fractional-delayed signal $s(n-\tau)$ pass through the designed FIR filter.

IV. NUMERICAL EXAMPLES

In this section, three examples performed with MATLAB language is now presented to evaluate the performance of the proposed fractional delay filters. The generalized eigenvector corresponding to maximum eigenvalue can be found by using the MATLAB instruction EIG.

Example 1: Polynomial signal

In the first example, the signal $s(n)$ is chosen as a polynomial signal below:

$$s(n) = 0.2n + 0.005\,n^2 \tag{37}$$

Thus, its fractionally delayed signal can be easily computed by

$$s(n-\tau) = 0.2(n-\tau) + 0.005\,(n-\tau)^2 \tag{38}$$

Since the simulated white-noise signal $v(n)$ and polynomial signal $s(n)$ are known, the components $y_s(n)$ and $y_v(n)$ in the fractional delay filter output $y(n)$ can be computed by

$$y_s(n) = \sum_{k=0}^{N} h(k)s(n-k) \tag{39a}$$

$$y_v(n) = \sum_{k=0}^{N} h(k)v(n-k) \tag{39b}$$

In order to evaluate the performances of various design methods, two performance indices are defined below:

(1) If $H(z)$ is a good fractional delay filter, the output $y(n)$ in (36) will be very close to $s(n-\tau)$. So, the absolute error between $y(n)$ and $s(n-\tau)$ can be used to evaluate whether $H(z)$ is a good fractional delay filter or not. The error is defined by

$$Err = \frac{1}{L-N}\sum_{n=N+1}^{L}\left|y(n)-s(n-\tau)\right| \tag{40}$$

Obviously, the smaller Err is, the better fractional delay filter $H(z)$ has.

(2) It is desirable that the SNR at fractional delay filter output is as high as possible. Using the time-average method, the signal power P_s and noise power P_v at the filter output can be estimated by

$$P_s = \frac{1}{L-N}\sum_{n=N+1}^{L}\left|y_s(n)\right|^2 \tag{41a}$$

$$P_v = \frac{1}{L-N}\sum_{n=N+1}^{L}\left|y_v(n)\right|^2 \tag{41b}$$

Based on the above results, the SNR of the output signal of fractional delay filter is estimated by

$$SNR_o = \frac{P_s}{P_v} \tag{42}$$

Obviously, the higher SNR_o is, the better fractional delay filter $H(z)$ has.

Here, the proposed method will be compared with two conventional fractional delay filters listed below:

Method 1: Lagrange maximally flat fractional delay filter whose filter coefficients are given by equation (29).

Method 2: Hamming-window based fractional delay filter whose filter coefficients are given by

$$h(k) = w(k)h_i(k) \tag{43}$$

with

$$h_i(k) = \frac{\sin(\pi(k-\tau))}{\pi(k-\tau)} \tag{44a}$$

$$w(k) = 0.54 - 0.46\cos\left(\tfrac{2\pi k}{N}\right) \tag{44b}$$

Note that the above two methods are not signal-dependent approaches. Now, let us compare the proposed method with two conventional methods by using Err and SNR_o. The parameters are chosen as noise variance $\sigma_v^2 = 36$, filter order $N = 20$, length of input signal $L = 200$, frequency point $\omega_0 = 0.01\pi$, number of derivative constraints $M = 5$ and $\eta = 0.0001$. Fig.1 shows the magnitude responses and group delay responses of three designed fractional delay filters. The solid line is actual response and dashed line is ideal response. The magnitude response of the proposed method is small at high-frequency band for attenuating the corrupted noise. Due to the small magnitude at high-frequency band, the large group delay error at high-frequency band is allowable. Fig.2 shows the corresponding output signals $y(n)$ of the designed fractional delay filters. Clearly, the outputs of Lagrange design and Hamming-window based design are noisy, so they are not good for fractionally delaying a noisy signal. Table I lists the Err and SNR_o of three design methods. From these results, it is clear that the proposed method is better than two conventional methods because Err is the smallest and SNR_o is the highest among them.

TABLE I
PERFORMANCE COMPARISON OF THREE DESIGN METHODS OF FRACTIONAL DELAY FILTERS IN EXAMPLE 1.

Design methods	*Err*	*SNR*o
Lagrange method	4.3315	371.71
Hamming-window method	4.4685	335.99
Proposed method	2.9339	840.18

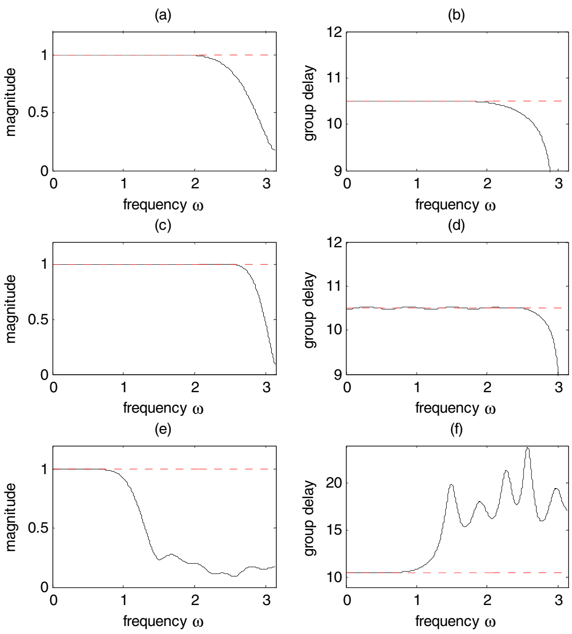

Fig.1 The magnitude responses and group delay responses of the designed fractional delay filters in example 1. The solid line is actual response and dashed line is ideal response. (a)(b) The results of Lagrange method. (c)(d) The results of Hamming-window based method. (e)(f) The results of the proposed method.

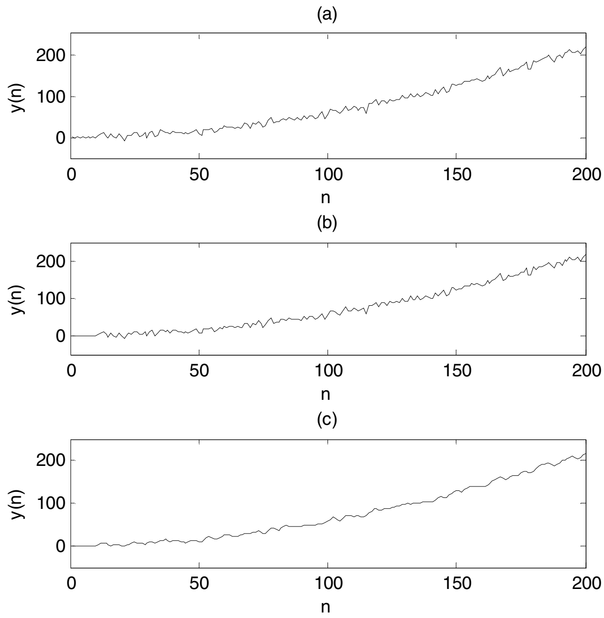

Fig.2 The output signals $y(n)$ of the designed fractional delay filters in example 1. (a) The output of Lagrange method. (b) The output of Hamming-window based method. (c) The output of the proposed method.

Example 2: Single sinusoidal signal

In the second example, the signal $s(n)$ is chosen as a single sinusoidal signal below:

$$s(n) = 10\ \sin(\ 0.1\pi n) \tag{45}$$

Thus, its fractionally delayed signal can be computed by

$$s(n-\tau) = 10\ \sin(\ 0.1\pi(n-\tau)) \tag{46}$$

Here, the Err and SNR_o in equations (40) and (42) are also used to evaluate the performance for single sinusoidal signal case. The parameters are chosen as noise variance $\sigma_v^2 = 25$, filter order $N = 20$, length of input signal $L = 200$, frequency point $\omega_0 = 0.1\pi$, number of derivative constraints $M = 3$ and $\eta = 0.0001$. Fig.3 depicts the magnitude responses and group delay responses of three designed fractional delay filters. The magnitude response of the proposed method is small at high-frequency band for attenuating the corrupted noise. Due to the small magnitude at high-frequency band, the large group delay error at high-frequency band is allowable. Fig.4 depicts the corresponding output signals $y(n)$ of the designed fractional delay filters. Clearly, the outputs of Lagrange design and Hamming-window based design are also noisy. Table II lists the Err and SNR_o of the three design methods. From these results, it is clear that the proposed method is better than two conventional methods in this case because Err is the smallest and SNR_o is the highest among them.

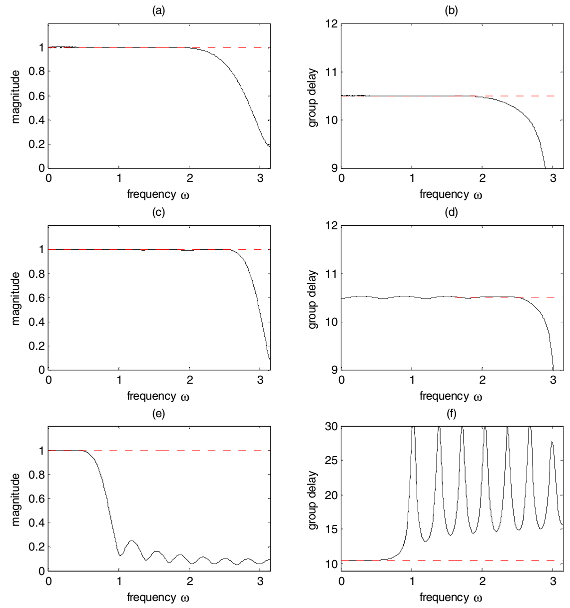

Fig.3 The magnitude responses and group delay responses of the designed fractional delay filters in example 2. The solid line is actual response and dashed line is ideal response. (a)(b) The results of Lagrange method. (c)(d) The results of Hamming-window based method. (e)(f) The results of the proposed method.

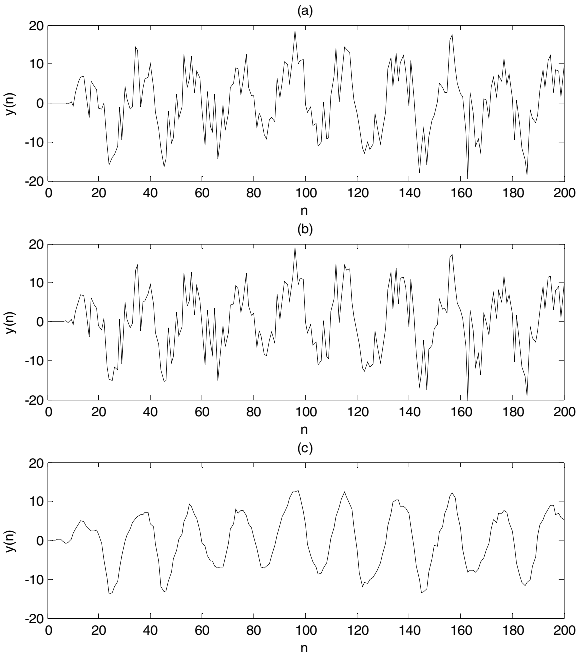

Fig.4 The output signals $y(n)$ of the designed fractional delay filters in example 2. (a) The output of Lagrange method. (b) The output of Hamming-window based method. (c) The output of the proposed method.

TABLE II
PERFORMANCE COMPARISON OF THREE DESIGN METHODS OF FRACTIONAL DELAY FILTERS IN EXAMPLE 2.

Design methods	*Err*	*SNR*o
Lagrange method	3.3558	2.7442
Hamming-window method	3.5212	2.4134
Proposed method	1.9185	8.6781

Example 3: Multiple sinusoidal signals

In the third example, the signal $s(n)$ is chosen as the multiple sinusoidal signals below

$$s(n) = 5\sin(0.2\pi n) + 5\cos(0.3\pi n) \quad (47)$$

Thus, its fractionally delayed signal can be computed by

$$\begin{aligned} s(n-\tau) &= 5\sin(0.2\pi(n-\tau)) \\ &\quad + 5\cos(0.3\pi(n-\tau)) \end{aligned} \quad (48)$$

Here, the parameters are chosen as $\sigma_v^2 = 25$, $N = 20$, $L = 200$, $\omega_0 = 0.25\pi$, $M = 4$ and $\eta = 0.0001$. Fig.5 shows the magnitude responses and group delay responses of the designed fractional delay filters. Fig.6 shows the corresponding output signals $y(n)$ of the designed fractional delay filters. Clearly, the outputs of Lagrange design and Hamming-window based design are also noisy. Table III lists the Err and SNR_o of the three design methods. From these results, it is clear that the proposed method is better than two conventional methods in this case.

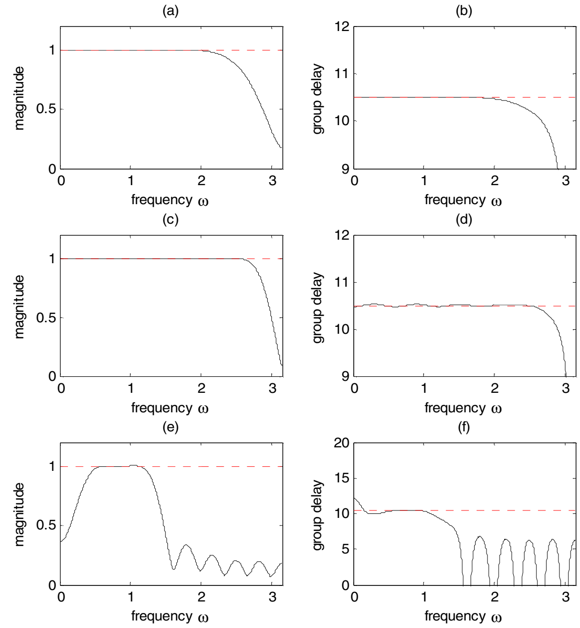

Fig.5 The magnitude responses and group delay responses of the designed fractional delay filters in example 3. The solid line is actual response and dashed line is ideal response. (a)(b) The results of Lagrange method. (c)(d) The results of Hamming-window based method. (e)(f) The results of the proposed method.

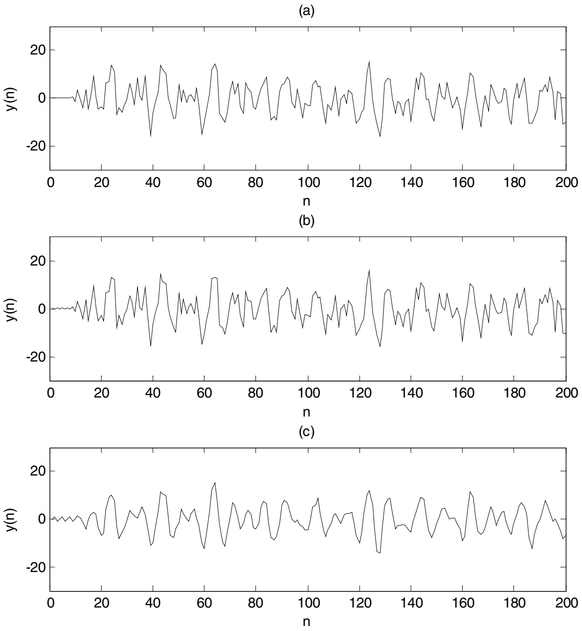

Fig.6 The output signals $y(n)$ of the designed fractional delay filters in example 3. (a) The output of Lagrange method. (b) The output of Hamming-window based method. (c) The output of the proposed method.

TABLE III
PERFORMANCE COMPARISON OF THREE DESIGN METHODS OF FRACTIONAL DELAY FILTERS IN EXAMPLE 3.

Design methods	*Err*	*SNR*o
Lagrange method	3.6069	1.2089
Hamming-window method	3.7684	1.0844
Proposed method	2.3335	3.0925

V. CONCLUSIONS

In this paper, a new approach to the design of digital FIR fractional delay filter with consideration of noise attenuation has been presented. The design is based on the maximization of SNR at the output of the fractional delay filter under the constraint that actual frequency response and ideal response have several same derivatives at the prescribed frequency point. The optimal filter coefficients are easily obtained from the generalized eigenvector associate with maximum eigenvalue of a pair of matrices. Numerical examples have shown the effectiveness of the proposed method. However, only one-dimensional filter is studied here. Thus, it is interesting to extend the proposed method to two-dimensional filter case. This topic will be investigated in the future.

REFERENCES

[1] S. Das and I. Pan, *Fractional Order Signal Processing: Introductory Concepts and Applications*, Springer, 2012.

[2] H. Sheng, Y.Q. Chen and T.S. Qiu, *Fractional Processes and Fractional-Order Signal Processing: Techniques and Applications*, Springer, 2012.

[3] R. Herrmann, *Fractional Calculus: An Introduction for Physicists*, World Scientific, 2011.

[4] M.D. Ortigueira, *Fractional Calculus for Scientists and Engineers*, Springer, 2011.

[5] T. I. Laakso, V. Valimaki, M. Karjalainen and U.K. Laine, "Splitting the unit delay: tool for fractional delay filter design," *IEEE Signal Processing Magazine*, pp.30-60, Jan. 1996.

[6] S.C. Pei and C.C. Tseng, "A comb filter design using fractional sample delay," *IEEE Trans. on Circuits and Systems-II: Analog and Digital Signal Processing,* vol.45, pp.649-653, June 1998.

[7] S.C. Pei, P.H. Wang and H.S. Lin, "Closed-form design of maximally flat FIR fractional delay filter," *IEEE Signal Processing Letters*, vol.13, pp.405-408, July 2006.

[8] C.C. Tseng and S.L. Lee, "Design of fractional delay filter using discrete Fourier transform interpolation method," *Signal Processing*, vol.90, pp.1313-1322, April, 2010.

[9] T.B. Deng, "Symmetric structures for odd-order maximally flat and weighted-least-squares variable fractional-delay filters," *IEEE Trans. on Circuits and Systems-I: Regular Papers,* vol.54, pp.2718-2732, Dec. 2007.

[10] J.J Shyu, S.C. Pei, C.H. Chan and Y.D. Huang, "Minimax design of variable fractional-delay FIR digital filters by iterative weighted least-squares approach," *IEEE Signal Processing Letters*, vol.15, pp.693-696, 2008.

[11] T.B. Deng, "Decoupling minimax design of low-complexity variable fractional-delay FIR digital filters," *IEEE Trans. on Circuits and Systems-I: Regular Papers,* vol.58, pp.2398-2408, Oct. 2011.

[12] T.B. Deng, "Minimax design of low-complexity even-order variable fractional-delay filters using second-order cone programming," *IEEE Trans. on Circuits and Systems-II: Express Briefs,* vol.58, pp.692-696, Oct. 2011.

[13] S.C. Pei and C.C. Tseng, "A new eigenfilter based on total least squares error criterion," *IEEE Trans. on Circuits and Systems-I: Fundamental Theory and Applications,* vol.48, pp.699-709, June 2001.