On L_2-Sensitivity of State-Space Digital Filters under Gramian-Preserving Frequncy Transformation

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***Abstract*—This paper aims to reveal the relationship between the minimum L_2-sensitivity of state-space digital filters and the Gramian-preserving frequency transformation. To this end, we first give a prototype low-pass state-space filter in such a manner that its structure becomes the minimum L_2-sensitivity structure. Then we apply the Gramian-preserving (LP-LP) frequency transformation with a tunable parameter to this prototype filter. In this way we obtain a low-pass state-space filter with tunable cutoff frequency from a prescribed prototype low-pass filter with minimum L_2-sensitivity. For this tunable low-pass state-space filter, we evaluate the L_2-sensitivity over the entire range of cutoff frequencies. The evaluation result shows that, although the minimality of the L_2-sensitivity is not preserved under the frequency transformation, the L_2-sensitivity of the tunable filter given in this way becomes very close to the minimum value for arbitrary cutoff frequencies.**

I. Introduction

It is very important to consider the finite wordlength problem of digital filters because a transfer function with finite wordlength is affected by the quantization and thus always deviates from the corresponding ideal transfer function. This problem has been extensively studied in the literature of digital filters, and many elegant methods have been proposed for synthesis of digital filter structures that are optimal with respect to this problem. Among such methods, minimization of L_2-sensitivity [1–11] is one of the primary theoretical methods for synthesis of optimal structures. The L_2-sensitivity is known as an evaluation function that measures the aforementioned deviation of a transfer function, and minimization of this sensitivity leads to the optimal filter structures that achieve high accuracy with respect to the coefficient quantization.

Although the minimization of the L_2-sensitivity can be applied to any kind of linear transfer function, the minimization problem is a non-linear one and thus very difficult to solve. Therefore, solving this minimization problem usually relies on iterative numerical calculations that involve both theoretically and numerically complicated algebraic manipulations. Hence, in real-time applications dealing with time-varying transfer functions such as variable digital filters and adaptive digital filters, it is usually difficult to use the L_2-sensitivity minimization scheme because the minimization process requires very high computational cost and may be unable to keep up with the sampling process.

The aim of our research is to easily realize time-varying digital filters that guarantee high performance with respect to the L_2-sensitivity. In order to achieve this goal, we need to establish a simple theoretical framework that can easily deal with time-varying filters under the requirement that the corresponding filter structures are optimal or nearly optimal with respect to the L_2-sensitivity. In this paper, we pay attention to the Gramian-preserving frequency transformation [12–14] as such a framework. The Gramian-preserving frequency transformation is an extention of the Constantinides frequency transformation [15] to state-space-based digital filters. As is well-known, the Constantinides frequency transformation is a simple theoretical tool that enables us to convert a given prototype IIR low-pass filter into other types of filter such as low-pass filter of different cutoff frequency, high-pass filter, band-pass filter, and band-stop filter. This simple conversion tool is very suitable to realization of time-varying filters and thus has brought many attractive methods for design of variable digital filters [16–20] and adaptive digital filters [21–24]. In addition to this simplicity, the Gramian-preserving frequency transformation provides further benefit that the performance of filter structures with respect to the aforementioned finite wordlength problem can be preserved. In other words, given a prototype low-pass filter of which structure optimizes an evaluation function with respect to the finite wordlength problem, the Gramian-preserving frequency transformation can convert such a prototype filter into other filters of which structures are also optimal. Therefore, the Gramian-preserving frequency transformation is a promising method for design and synthesis of time-varying filters with high performance structures. In fact, some applications of this transformation to high-performance variable digital filters have been already proposed [25], [26]. It should be noted here, however, the property of preserving such high-performance is guaranteed for not the L_2-sensitivity, but the other types of evaluation function such as roundoff noise and L_1/L_2-mixed sensitivity. That is, it is still unclear whether the optimality of the L_2-sensitivity is also preserved under the Gramian-preserving frequency transformation. This background motivates the present work.

In this paper, we attempt to reveal the relationship between the Gramian-preserving frequency transformation and the L_2-sensitivity. Compared with other evaluation functions such as roundoff noise and L_1/L_2-mixed sensitivity, the L_2-sensitivity is much more difficult to analyze because of its complicated formulation. Therefore, in this paper we reveal the relationship between the Gramian-preserving frequency transformation and the L_2-sensitivity by numerical experiments rather than a mathematical analysis. Also, for simplicity the Gramian-preserving frequency transformation is restricted to the low-pass-low-pass (LP-LP) transformation. That is, we prepare a prototype low-pass filter with the minimum L_2-sensitivity structure, and we investigate the L_2-sensitivity under the change of the cutoff frequency of the low-pass filter. Our main result shows that, the minimality of the L_2-sensitivity is not preserved under the Gramian-preserving frequency transformation, but any of the transformed filters has the L_2-sensitivity very close to the minimum value.

II. Preliminaries

This section reviews the L_2-sensitivity and the Gramian-preserving frequency transformation.

A. L_2-sensitivity

The L_2-sensitivity of a linear discrete-time transfer function is described in terms of the state-space formulation. Given an N-th order stable digital filter, the input-output relationship can be described by the following state-space representation

$$\begin{aligned} \boldsymbol{x}(n+1) &= \boldsymbol{A}\boldsymbol{x}(n) + \boldsymbol{b}u(n) && (1)\\ y(n) &= \boldsymbol{c}\boldsymbol{x}(n) + du(n) && (2) \end{aligned}$$

where $u(n)$ and $y(n)$ are the scalar input and the scalar output, $\boldsymbol{x}(n) \in \Re^{N\times 1}$ is the state vector, and $\boldsymbol{A} \in \Re^{N\times N}$, $\boldsymbol{b} \in \Re^{N\times 1}$, $\boldsymbol{c} \in \Re^{1\times N}$ and $d \in \Re^{1\times 1}$ are real-valued coefficients. The transfer function $H(z)$ of this state-space system is given in terms of $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, d)$ by

$$H(z) = d + \boldsymbol{c}(z\boldsymbol{I}_N - \boldsymbol{A})^{-1}\boldsymbol{b} \tag{3}$$

where $\boldsymbol{I}_N$ denotes the $N \times N$ identity matrix. Throughout this paper, we assume that the state-space filter is asymptotically stable (i.e. the matrix $\boldsymbol{A}$ has all eigenvalues inside the unit circle), and that the state-space filter is controllable and observable. Note that the set $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ is non-unique for a given transfer function $H(z)$. This fact means that the set $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ depends upon the configuration of filter structures.

The L_2-sensitivity of an N-th order transfer function $H(z)$ is defined as the following evaluation function $S(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$:

$$S(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}) \equiv \left\| \frac{\partial H(z)}{\partial \boldsymbol{A}} \right\|_2^2 + \left\| \frac{\partial H(z)}{\partial \boldsymbol{b}} \right\|_2^2 + \left\| \frac{\partial H(z)}{\partial \boldsymbol{c}} \right\|_2^2. \tag{4}$$

The above-defined L_2-sensitivity can be explicitly described by the following representation [2]:

$$\begin{aligned} S(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}) &= \mathrm{tr}(\boldsymbol{K})\mathrm{tr}(\boldsymbol{W}) + \mathrm{tr}(\boldsymbol{K}) + \mathrm{tr}(\boldsymbol{W}) \\ &\quad +2\sum_{i=1}^{\infty} \mathrm{tr}(\boldsymbol{K}_i)\mathrm{tr}(\boldsymbol{W}_i) && (5) \end{aligned}$$

where $\boldsymbol{K}$ and $\boldsymbol{W}$ are the controllability/observability Gramians, and $\boldsymbol{K}_i$ and $\boldsymbol{W}_i$ for $i \geq 1$ are called the general controllability/observability Gramians. The controllability Gramian $\boldsymbol{K}$ and the observability Gramian $\boldsymbol{W}$ are the positive definite matrices given as the solutions to the following Lyapunov equations

$$\begin{aligned} \boldsymbol{K} &= \boldsymbol{A}\boldsymbol{K}\boldsymbol{A}^T + \boldsymbol{b}\boldsymbol{b}^T && (6)\\ \boldsymbol{W} &= \boldsymbol{A}^T\boldsymbol{W}\boldsymbol{A} + \boldsymbol{c}^T\boldsymbol{c} && (7) \end{aligned}$$

and the general Gramians are simply calculated from $\boldsymbol{A}$, $\boldsymbol{K}$ and $\boldsymbol{W}$ as [5]

$$\begin{aligned} \boldsymbol{K}_i &= \frac{1}{2}\left(\boldsymbol{A}^i\boldsymbol{K} + \boldsymbol{K}(\boldsymbol{A}^T)^i\right) && (8)\\ \boldsymbol{W}_i &= \frac{1}{2}\left((\boldsymbol{A}^T)^i\boldsymbol{W} + \boldsymbol{W}\boldsymbol{A}^i\right). && (9) \end{aligned}$$

Since the set $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ depends upon the configuration of filter structures, all of those Gramians and the value of $S(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ also depend upon filter structures. Hence the minimization of the L_2-sensitivity is achieved by appropriate choice of filter structures.

Finally, it should be noted that the controllability Gramian $\boldsymbol{K}$ and the observability Gramian $\boldsymbol{W}$ are especially important because these two Gramians play central roles in optimization of not only the L_2-sensitivity but also other evaluation functions (e.g. roundoff noise and L_1/L_2-mixed sensitivity). For details, see [27–32] and the references therein.

B. Gramian-preserving frequency transformation

As stated in Section I, in this paper the frequency transformation is restricted to the LP-LP transformation. The conventional LP-LP transformation proposed by Constantinides [15] is described as follows. First, we prepare an N-th order prototype low-pass filter with the transfer function denoted by $H_{\mathrm{p}}(z)$. Then we apply the following substitution

$$\begin{aligned} H_{\mathrm{d}}(z,\eta) &= H_{\mathrm{p}}(z)|_{z^{-1}\leftarrow T(z)} \\ T(z) &= \frac{z^{-1}-\eta}{1-\eta z^{-1}} && (10) \end{aligned}$$

which results in the desired N-th order low-pass transfer function $H_{\mathrm{d}}(z,\eta)$ of which cutoff frequency is different from that of $H_{\mathrm{p}}(z)$ and controlled by the parameter η. This parameter is related to the cutoff frequencies of these two filters as

$$\eta = \frac{\sin\left(\frac{\omega_{\mathrm{p}}-\omega_{\mathrm{d}}}{2}\right)}{\sin\left(\frac{\omega_{\mathrm{p}}+\omega_{\mathrm{d}}}{2}\right)}, \quad |\eta| < 1 \tag{11}$$

where ω_{p} and ω_{d} respectively denote the cutoff frequencies of $H_{\mathrm{p}}(z)$ and $H_{\mathrm{d}}(z,\eta)$.

Now we introduce the Gramian-preserving frequency transformation [12–14]. Let $(\boldsymbol{A}_{\mathrm{p}}, \boldsymbol{b}_{\mathrm{p}}, \boldsymbol{c}_{\mathrm{p}}, d_{\mathrm{p}})$ be a state-space representation of the prototype filter $H_{\mathrm{p}}(z)$. Then, the Gramian-preserving frequency transformation in the LP-LP case converts this state-space filter into the following set

$(\boldsymbol{A}_\mathrm{d}, \boldsymbol{b}_\mathrm{d}, \boldsymbol{c}_\mathrm{d}, d_\mathrm{d})$[1]

$$\begin{aligned} \boldsymbol{A}_\mathrm{d} &= (\eta \boldsymbol{I}_N + \boldsymbol{A}_\mathrm{p})(\boldsymbol{I}_N + \eta \boldsymbol{A}_\mathrm{p})^{-1} \\ \boldsymbol{b}_\mathrm{d} &= \sqrt{1-\eta^2}(\boldsymbol{I}_N + \eta \boldsymbol{A}_\mathrm{p})^{-1}\boldsymbol{b}_\mathrm{p} \\ \boldsymbol{c}_\mathrm{d} &= \sqrt{1-\eta^2}\boldsymbol{c}_\mathrm{p}(\boldsymbol{I}_N + \eta \boldsymbol{A}_\mathrm{p})^{-1} \\ d_\mathrm{d} &= d_\mathrm{p} - \eta \boldsymbol{c}_\mathrm{p}(\boldsymbol{I}_N + \eta \boldsymbol{A}_\mathrm{p})^{-1}\boldsymbol{b}_\mathrm{p} \end{aligned} \quad (12)$$

and the transfer function corresponding to (12) is the same as $H_d(z, \eta)$ shown in (10).

An important property of the transformation (12) is that the controllability/observability Gramians are preserved, i.e. the following relationship holds for any η:

$$\begin{aligned} \boldsymbol{K}_\mathrm{d} &= \boldsymbol{K}_\mathrm{p} \\ \boldsymbol{W}_\mathrm{d} &= \boldsymbol{W}_\mathrm{p} \end{aligned} \quad (13)$$

where $(\boldsymbol{K}_\mathrm{p}, \boldsymbol{W}_\mathrm{p})$ and $(\boldsymbol{K}_\mathrm{d}, \boldsymbol{W}_\mathrm{d})$ are the controllability and observability Gramians of $(\boldsymbol{A}_\mathrm{p}, \boldsymbol{b}_\mathrm{p}, \boldsymbol{c}_\mathrm{p}, d_\mathrm{p})$ and $(\boldsymbol{A}_\mathrm{d}, \boldsymbol{b}_\mathrm{d}, \boldsymbol{c}_\mathrm{d}, d_\mathrm{d})$, respectively.

III. Main Result

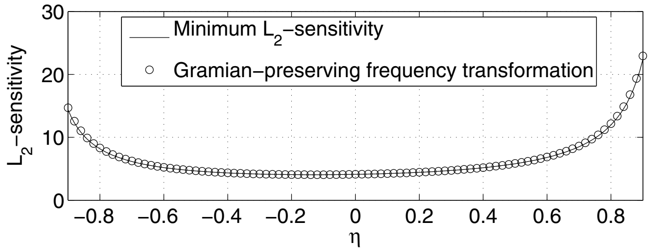
Before presenting our main result, we first adress the details of the problem statement. In the case of other evaluation functions such as roundoff noise and L_1/L_2-mixed sensitivity, the following facts hold [12–14]:

- If the structure of a prototype state-space filter $(\boldsymbol{A}_\mathrm{p}, \boldsymbol{b}_\mathrm{p}, \boldsymbol{c}_\mathrm{p}, d_\mathrm{p})$ is optimal with respect to roundoff noise, the transformed filters $(\boldsymbol{A}_\mathrm{d}, \boldsymbol{b}_\mathrm{d}, \boldsymbol{c}_\mathrm{d}, d_\mathrm{d})$ given by (12) are also optimal for any η.
- If the structure of a prototype state-space filter $(\boldsymbol{A}_\mathrm{p}, \boldsymbol{b}_\mathrm{p}, \boldsymbol{c}_\mathrm{p}, d_\mathrm{p})$ is optimal with respect to L_1/L_2-mixed sensitivity, the transformed filters $(\boldsymbol{A}_\mathrm{d}, \boldsymbol{b}_\mathrm{d}, \boldsymbol{c}_\mathrm{d}, d_\mathrm{d})$ given by (12) are also optimal for any η.

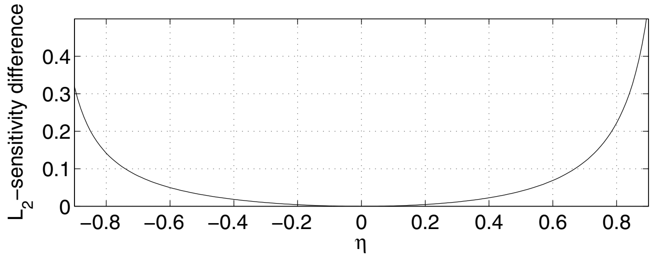
These facts can be proved from the Gramian-preserving property (13) and the fact that these evaluation functions are characterized by only the controllability/observability Gramians. On the other hand, in the case of the L_2-sensitivity, the evaluation function (5) is related to not only the controllability/observability Gramians but also the general Gramians $\boldsymbol{K}_i$ and $\boldsymbol{W}_i$. In addition, unfortunately these general Gramians are not necessarily preserved under our transformation (12). Because of these facts, up to the present it has not been clarified how the L_2-sensitivity behaves under the Gramian-preserving frequency transformation.

In the sequel we reveal the relationship between the Gramian-preserving LP-LP transformation (12) and the L_2-sensitivity. To be specific, we numerically show this relationship in two cases. The first case is the L_2-sensitivity minimization problem without any constraint, and the second case is the minimization problem with the dynamic-range scaling constraint. These two cases are widely investigated in the literature [1–11].

[1]To be precise, the Gramian-preserving frequency transformation in the LP-LP case (12) is originally proposed by Mullis and Roberts [33]. Our work [12–14] is an extension of (12) to the general frequency transformations including LP-HP, LP-BP, LP-BS, and analog frequency transformations.



(a)



(b)

Fig. 1. Behavior of the L_2-sensitivity without any constraint: (a) L_2-sensitivity for the optimal structure and the structure given by the Gramian-preserving frequency transformation, and (b) difference of the L_2-sensitivity between the two structures.

A. Case 1: L_2-sensitivity without constraint

Consider a prototype low-pass filter with the following transfer function

$$H_\mathrm{p}(z) = \frac{0.1667 + 0.5000z^{-1} + 0.5000z^{-2} + 0.1667z^{-3}}{1 + 0.3333z^{-2}}. \quad (14)$$

This prototype filter is the third-order Butterworth low-pass filter with the cutoff frequency of 0.5π rad. Applying the L_2-sensitivity minimization to this prototype filter, we obtain the following state-space representation

$$\left(\begin{array}{c|c} \boldsymbol{A}_\mathrm{p} & \boldsymbol{b}_\mathrm{p} \\ \hline \boldsymbol{c}_\mathrm{p} & d_\mathrm{p} \end{array}\right) = \left(\begin{array}{ccc|c} 0.2255 & -0.3677 & 0.0791 & 0.7945 \\ 0.7846 & -0.0335 & -0.2463 & 0.1358 \\ 0.0867 & 0.3900 & -0.1920 & 0.0839 \\ \hline 0.5032 & 0.5849 & 0.2477 & 0.1667 \end{array}\right) \quad (15)$$

which is optimal with respect to the evaluation function shown in (4) and (5). The value of the L_2-sensitivity of this state-space filter (i.e. the minimum value of the L_2-sensitivity for the transfer function (14)) is calculated to be 4.1065.

We are now ready to show the behavior of the L_2-sensitivity under the Gramian-preserving LP-LP transformation. We apply (12) to (15) and obtain a state-space filter $(\boldsymbol{A}_\mathrm{d}, \boldsymbol{b}_\mathrm{d}, \boldsymbol{c}_\mathrm{d}, d_\mathrm{d})$ of which cutoff frequency is determined by a given parameter η. Figure 1(a) shows the resultant L_2-

sensitivity of $(\boldsymbol{A}_{\rm d}, \boldsymbol{b}_{\rm d}, \boldsymbol{c}_{\rm d}, d_{\rm d})$ versus the parameter η, together with the corresponding optimal (minimum) value of the L_2-sensitivity. Also, the difference between the L_2-sensitivity of $(\boldsymbol{A}_{\rm d}, \boldsymbol{b}_{\rm d}, \boldsymbol{c}_{\rm d}, d_{\rm d})$ and the minimum L_2-sensitivity is shown in Fig. 1(b). From this result we first see that the Gramian-preserving frequency transformation unfortunately does not necessarily preserve the optimal structure. That is, although the prototype state-space filter (15) is given as the optimal structure with respect to the L_2-sensitivity, the transformed state-space filters given by (12) are not necessarily optimal. However, Fig. 1 also shows that the L_2-sensitivity of the transformed state-space filters is very close to the minimum L_2-sensitivity for arbitrary values of η. Therefore we emphasize that the Gramian-preserving frequency transformation is useful enough for design and synthesis of high-performance digital filters with respect to the L_2-sensitivity, as well as other evaluation functions: once we prepare a prototype state-space filter with minimum L_2-sensitivity, the Gramian-preserving frequency transformation allows us to easily convert this prototype filter into other filters of which L_2-sensitivity becomes very close to the minimum value.

B. Case 2: L_2-sensitivity with dynamic-range scaling constraint

Here we discuss the case of the L_2-sensitivity with the dynamic-range scaling constraint. In this case, the controllability Gramian $\boldsymbol{K}$ of an N-th order state-space filter $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, d)$ must satisfy the following relationship

$$K_{ii} = 1, \quad 1 \le i \le N \tag{16}$$

where K_{ii} denotes the i-th diagonal entry of $\boldsymbol{K}$. This constraint is practically important for suppression of the overflow in the state vector of a state-space filter.

The prototype filter to be used here is the same as (14). The optimal state-space representation $(\boldsymbol{A}_{\rm p}, \boldsymbol{b}_{\rm p}, \boldsymbol{c}_{\rm p}, d_{\rm p})$ with respect to the L_2-sensitivity subject to the constraint (16) is given as follows:

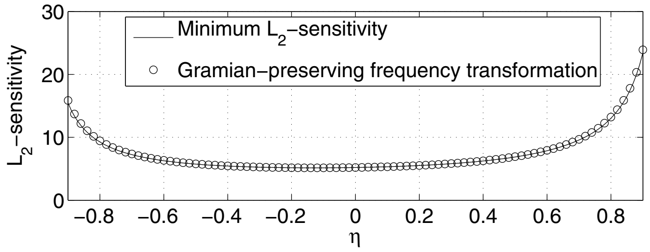
$$\left(\begin{array}{c|c} \boldsymbol{A}_{\rm p} & \boldsymbol{b}_{\rm p} \\ \hline \boldsymbol{c}_{\rm p} & d_{\rm p} \end{array}\right) = \left(\begin{array}{ccc|c} 0.0903 & 0.2952 & 0.1379 & 0.9075 \\ 0.0696 & -0.0594 & -0.5401 & 0.8642 \\ 0.2913 & 0.7413 & -0.0309 & 0.3113 \\ \hline 0.3676 & 0.0658 & 0.3519 & 0.1667 \end{array}\right). \tag{17}$$

The corresponding controllability Gramian $\boldsymbol{K}_{\rm p}$ becomes

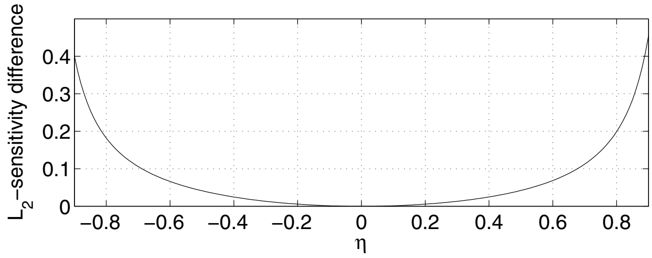
$$\boldsymbol{K}_{\rm p} = \begin{pmatrix} 1.0000 & 0.6611 & 0.6611 \\ 0.6611 & 1.0000 & 0.1281 \\ 0.6611 & 0.1281 & 1.0000 \end{pmatrix} \tag{18}$$

which satisfies the relationship (16). The L_2-sensitivity of this state-space filter is calculated to be 5.1972, which is the minimum value for the transfer function (14) subject to the dynamic-range scaling constraint.

We now show the behavior of the L_2-sensitivity under the Gramian-preserving LP-LP transformation by taking the same procedure as in the previous case. Figure 2 shows the result.



(a)



(b)

Fig. 2. Behavior of the L_2-sensitivity with dynamic-range scaling constraint: (a) L_2-sensitivity for the optimal structure and the structure given by the Gramian-preserving frequency transformation, and (b) difference of the L_2-sensitivity between the two structures.

Here, note that the transformed filters given by (12) satisfy the constraint (16) for any η because the Gramian-preserving frequency transformation keeps $K_{\rm p}$ invariant regardless of the value of η. Now, we see that Fig. 2 is very similar to the previous case, and thus we obtain the same conclusion as in the previous case: The Gramian-preserving frequency transformation does not necessarily preserve the optimal L_2-sensitivity, but this transformation is still useful because any of the transformed filters has the L_2-sensitivity very close to the corresponding minimum value.

IV. Conclusion

This paper has clarified the behavior of the L_2-sensitivity under the Gramian-preserving LP-LP transformation. Given a prototype state-space filter with the minimum L_2-sensitivity, we have applied the Gramian-preserving frequency transformation to this filter and have numerically investigated the L_2-sensitivity of the transformed filter. As a result, it has been confirmed that the transformed filters are not necessarily optimal with respect to the L_2-sensitivity, but that any transformed structure is found to be very close to the optimal structure. From this result we conclude that the Gramian-preserving frequency transformation is very useful in designing and synthesizing high-performance digital filters with respect to the L_2-sensitivity.

Since this result is numerically obtained, it is necessary to give a theoretical proof of this result. The proof is a very difficult task and thus remains as a future work.

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