On the Limit Cycles in the Minimum L_2-Sensitivity Realizations Subject to L_2-Scaling Constraints of Second-Order Digital Filters

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***Abstract*—This paper gives conjecture on the absence of limit cycles of the minimum L_2-sensitivity realizations subject to L_2-scaling constraints for second-order digital filters. We design second-order digital filters with various pole-zero configurations, synthesize the minimum L_2-sensitivity realizations subject to L_2-scaling constraints, and examine if their coefficient matrices satisfy a sufficient condition for the absence of limit cycles. As a result, in the range of practical pole radii, it is shown that the minimum L_2-sensitivity realizations subject to L_2-scaling constraints of second-order digital filters satisfy a sufficient condition for the absence of limit cycles. Furthermore, we demonstrate the absence of limit cycles of the minimum L_2-sensitivity realizations of a second-order digital filter by observing its zero-input response.**

I. Introduction

On the fixed-point implementation of digital filters, undesirable finite-word-length (FWL) effects arise. Limit cycles occur in recursive digital filters implemented with FWL due to the nonlinear action of adder overflow and quantization of the products.

The balanced realization and the minimum roundoff noise realization are the minimum L_1/L_2-sensitivity realizations *without* L_2-scaling constraints and *subject to* L_2-scaling constraints, respectively, and do not generate limit cycles [1], [2]. However, it would be more natural to use L_2-sensitivity than to use L_1/L_2-sensitivity as a coefficient sensitivity since L_2-sensitivity measure is formulated without any approximation while L_1/L_2-sensitivity is formulated with approximation [3]–[9]. Therefore, it is worth investigating the limit cycles of the minimum L_2-sensitivity realizations.

Our group previously proved the absence of limit cycles of the minimum L_2-sensitivity realizations *without* L_2-scaling constraints [10]. However, in actual signal processing scene, L_2-scaling constraints are necessary for preventing overflow of state variables. Hinamoto *et al.* proposed iterative solutions for synthesis of the minimum L_2-sensitivity realizations *subject to* L_2-scaling constraints [6], [7]. Our group proposed an iterative solution which achieves the L_2-sensitivity minimization *with quite fast convergence* [8]. Furthermore, our group proposed closed form solutions for second-order digital filters [9].

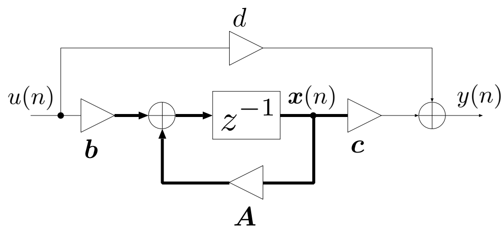


Fig. 1. Block diagram of a state-space digital filter.

In this paper, we numerically give conjecture on the absence of limit cycles of the minimum L_2-sensitivity realizations *subject to* L_2-scaling constraints of second-order digital filters. We design second-order digital filters for various pole-zero configurations, synthesize the minimum L_2-sensitivity realizations, and examine if their coefficient matrices satisfy a sufficient condition for the absence of limit cycles. Furthermore, we demonstrate the absence of limit cycles of the minimum L_2-sensitivity realizations *subject to* L_2-scaling constraints of a second-order digital filter by observing its zero-input response. As a result, we conjecture that the minimum L_2-sensitivity realizations *subject to* L_2-scaling constraints of second-order digital filters are free of limit cycles for most practical cases.

II. Preliminaries

A. State-Space Digital Filters

Consider a stable, controllable, and observable Nth-order state-space digital filter described by

$$\boldsymbol{x}(n+1) = \boldsymbol{A}\boldsymbol{x}(n) + \boldsymbol{b}u(n) \quad (1)$$

$$y(n) = \boldsymbol{c}\boldsymbol{x}(n) + du(n) \quad (2)$$

where $\boldsymbol{x}(n) \in \boldsymbol{R}^{N\times 1}$ is a state-vector, $u(n) \in \boldsymbol{R}$ is a scalar input, $y(n) \in \boldsymbol{R}$ is a scalar output, and $\boldsymbol{A} \in \boldsymbol{R}^{N\times N}$, $\boldsymbol{b} \in \boldsymbol{R}^{N\times 1}$, $\boldsymbol{c} \in \boldsymbol{R}^{1\times N}$, $d \in \boldsymbol{R}$ are coefficient matrices. The block diagram of the state-space digital filter $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, d)$ is shown in Fig. 1. The transfer function $H(z)$ is described in terms of the coefficient matrices $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, d)$ as $H(z) = \boldsymbol{c}(z\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{b}+d$.

B. L_2-Sensitivity

The L_2-sensitivity of the filter $H(z)$ with respect to the realization $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, d)$ is defined by

$$S(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}) = \left\| \frac{\partial H(z)}{\partial \boldsymbol{A}} \right\|_2^2 + \left\| \frac{\partial H(z)}{\partial \boldsymbol{b}} \right\|_2^2 + \left\| \frac{\partial H(z)}{\partial \boldsymbol{c}} \right\|_2^2 . \quad (3)$$

Hinamoto *et al.* expressed the L_2-sensitivity in terms of the general Gramians as follows [4]:

$$\begin{aligned} S(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}) &= \mathrm{tr}(\boldsymbol{W}_0)\mathrm{tr}(\boldsymbol{K}_0) + \mathrm{tr}(\boldsymbol{W}_0) + \mathrm{tr}(\boldsymbol{K}_0) \\ &\quad + 2\sum_{i=1}^{\infty} \mathrm{tr}(\boldsymbol{W}_i)\mathrm{tr}(\boldsymbol{K}_i) \end{aligned} \quad (4)$$

where $\boldsymbol{K}_0$ and $\boldsymbol{W}_0$ are the controllability and observability Gramians, respectively, $\boldsymbol{K}_i$ and $\boldsymbol{W}_i$ are the general controllability and observability Gramians, respectively. The controllability and observability Gramians $(\boldsymbol{K}_0, \boldsymbol{W}_0)$ can be obtained by solving the following Lyapunov equations:

$$\boldsymbol{K}_0 = \boldsymbol{A}\boldsymbol{K}_0\boldsymbol{A}^T + \boldsymbol{b}\boldsymbol{b}^T \quad (5)$$

$$\boldsymbol{W}_0 = \boldsymbol{A}^T\boldsymbol{W}_0\boldsymbol{A} + \boldsymbol{c}^T\boldsymbol{c}. \quad (6)$$

Our group previously proposed a novel expression of general Gramians as follows [8]:

$$\boldsymbol{K}_i = \frac{1}{2}\left(\boldsymbol{A}^i\boldsymbol{K}_0 + \boldsymbol{K}_0(\boldsymbol{A}^T)^i\right) \quad (7)$$

$$\boldsymbol{W}_i = \frac{1}{2}\left(\boldsymbol{W}_0\boldsymbol{A}^i + (\boldsymbol{A}^T)^i\boldsymbol{W}_0\right). \quad (8)$$

C. Limit Cycles in State-Space Digital Filters

Under zero input condition, the following state-space equations are obtained:

$$\boldsymbol{x}(n+1) = \boldsymbol{A}\boldsymbol{x}(n) \quad (9)$$

$$y(n) = \boldsymbol{c}\boldsymbol{x}(n) \quad (10)$$

by letting $u(n) = 0$ in (1) and (2). Eq. (9) describes the autonomous behavior of the state-space digital filter. In this equation, coefficient matrix $\boldsymbol{A}$ is called transition matrix. When this digital filter is stable, we have $\lim_{n\to\infty} \boldsymbol{x}(n) = \boldsymbol{0}$ for any initial state $\boldsymbol{x}(0)$. However, the actual digital filters implemented by finite word-length have nonlinearities due to adder overflow and quantization errors. For recursive digital filters, these nonlinearities cause undesirable *limit cycles*, which can be classified into *overflow limit cycles* and *granular limit cycles*. Adder overflow causes large-amplitude autonomous oscillations, which is called *overflow limit cycles*. On the other hand, quantization causes small-amplitude autonomous oscillations, which is called *granular limit cycles*.

The state transition of the digital filter considering the overflow is described by

$$\boldsymbol{x}(n+1) = f\left[\boldsymbol{A}\boldsymbol{x}(n)\right] \quad (11)$$

where f is a nonlinear function describing overflow characteristic. The nonlinear function f satisfies

$$|f_i(x_i)| \le |x_i|. \quad (12)$$

Overflow characteristics (two's complement, saturation, and zeroing) satisfy the above inequality. It is known that nonlinearity of the quantization using signed-magnitude truncation after addition is also described by the function f which satisfies the inequality (12).

D. Sufficient Conditions for the Absence of Limit Cycles

Under the conditions described by (11) and (12), Some sufficient conditions for state-space digital filters to be free of limit cycles have been proposed as follows:

Theorem 1: [11] If the transition matrix $\boldsymbol{A}$ satisfies

$$\|\boldsymbol{A}\|_2 < 1 \quad (13)$$

the system will be asymptotically stable, and no limit cycles can exist.

Theorem 2: [12] Let $\boldsymbol{A}$ be the transition matrix. If there is a diagonal matrix $\boldsymbol{D}$ with positive diagonal for which

$$\boldsymbol{D} - \boldsymbol{A}^T\boldsymbol{D}\boldsymbol{A} > 0 \quad (14)$$

is satisfied, the given system does not generate limit cycles. [1]

Theorem 3: [12] Let $\boldsymbol{A}$ be a 2×2 matrix whose eigenvalues satisfy $|\lambda| < 1$. There exists a positive definite diagonal matrix $\boldsymbol{D}$ which satisfies $\boldsymbol{D} - \boldsymbol{A}^T\boldsymbol{D}\boldsymbol{A} > 0$ if and only if

$$\begin{cases} \text{(a) } a_{12}a_{21} \ge 0 \\ \text{or} \\ \text{(b) } a_{12}a_{21} < 0,\ |a_{11} - a_{22}| + \det(\boldsymbol{A}) < 1 \end{cases} \quad (15)$$

for second-order digital filters.

Theorem 4: [1] The transition matrix $\boldsymbol{A}$ of an Nth-order state-space digital filter $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, d)$ satisfies $\boldsymbol{D} - \boldsymbol{A}^T\boldsymbol{D}\boldsymbol{A} > 0$ if the controllability Gramian $\boldsymbol{K}_0$ and observability Gramian $\boldsymbol{W}_0$ have the following relation:

$$\boldsymbol{W}_0 = \boldsymbol{D}\boldsymbol{K}_0\boldsymbol{D} \quad (16)$$

for a positive definite diagonal matrix $\boldsymbol{D}$.

If one of these sufficient conditions is satisfied for a given digital filter structure, we can conclude that the digital filter structure does not generate limit cycles. We have to note that limit cycles in state-space digital filters $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, d)$ only depend on coefficient matrix $\boldsymbol{A}$. The other coefficient matrices $(\boldsymbol{b}, \boldsymbol{c}, d)$ do not matter in considering limit cycles.

III. Limit Cycles in the Minimum L_2-Sensitivity Realizations

A. The L_2-Sensitivity Under Coordinate Transformations

Let $\boldsymbol{T}$ be a nonsingular $N \times N$ real matrix. If a coordinate transformation defined by $\bar{\boldsymbol{x}}(n) = \boldsymbol{T}^{-1}\boldsymbol{x}(n)$ is applied to a filter realization $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, d)$, we obtain a new realization which has the following coefficient matrices

$$(\bar{\boldsymbol{A}}, \bar{\boldsymbol{b}}, \bar{\boldsymbol{c}}, \bar{d}) = (\boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}, \boldsymbol{T}^{-1}\boldsymbol{b}, \boldsymbol{c}\boldsymbol{T}, d) \quad (17)$$

[1] In fact, it is known that $\|\boldsymbol{A}\|_2 < 1$ if and only if $\boldsymbol{I} - \boldsymbol{A}^T\boldsymbol{A} > 0$. Therefore, Theorem 1 is a special case of Theorem 2 where matrix $\boldsymbol{D}$ is chosen to be the identity matrix $\boldsymbol{I}$.

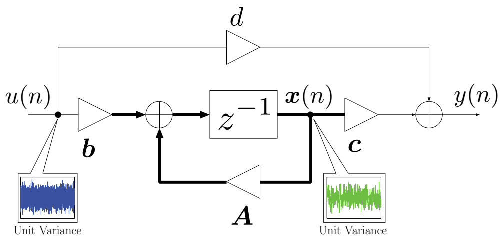


Fig. 2. L_2-scaling constraints of state-space digital filters.

and the following general Gramians

$$(\bar{\boldsymbol{K}}_i, \bar{\boldsymbol{W}}_i) = (\boldsymbol{T}^{-1}\boldsymbol{K}_i\boldsymbol{T}^{-T}, \boldsymbol{T}^T\boldsymbol{W}_i\boldsymbol{T}) \tag{18}$$

respectively. The L_2-sensitivity of the filter $(\bar{\boldsymbol{A}}, \bar{\boldsymbol{b}}, \bar{\boldsymbol{c}}, \bar{d})$ can be expressed as follows:

$$\begin{aligned} S(\boldsymbol{P}) = &\operatorname{tr}(\boldsymbol{W}_0\boldsymbol{P})\operatorname{tr}(\boldsymbol{K}_0\boldsymbol{P}^{-1}) + \operatorname{tr}(\boldsymbol{W}_0\boldsymbol{P}) + \operatorname{tr}(\boldsymbol{K}_0\boldsymbol{P}^{-1}) \\ &+2\sum_{i=1}^{\infty}\operatorname{tr}(\boldsymbol{W}_i\boldsymbol{P})\operatorname{tr}(\boldsymbol{K}_i\boldsymbol{P}^{-1}) \end{aligned} \tag{19}$$

where $\boldsymbol{P} = \boldsymbol{T}\boldsymbol{T}^T$ is a positive definite symmetric matrix.

B. Minimum L_2-Sensitivity Realizations Subject to L_2-scaling constraints

In order to prevent the overflow of state variables, the variance of state variables must be unity under the white Gaussian input with zero mean and unit variance as shown in Fig. 2. It follows that each diagonal element of the covariance matrix of $(\boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}, \boldsymbol{T}^{-1}\boldsymbol{b}, \boldsymbol{c}\boldsymbol{T}, d)$, which represents the variance of the state variables, must be unity such as

$$(\boldsymbol{T}^{-1}\boldsymbol{K}_0\boldsymbol{T}^{-T})_{ii} = 1 \ (i = 1, \cdots, N). \tag{20}$$

The above constraints are called L_2-*scaling constraints*. Under the constraints, L_2-sensitivity minimization problem *subject to* L_2-scaling constrains is formulated as follows [6]–[9]:

$$\begin{aligned} &\min_{\boldsymbol{P}} S(\boldsymbol{P}) \text{ in Eq. (19)} \\ &\text{subject to } \operatorname{tr}(\boldsymbol{K}_0\boldsymbol{P}^{-1}) = N. \end{aligned} \tag{21}$$

To this problem, Hinamoto *et al.* proposed iterative solutions [6], [7], and our group proposed an iterative solution *with quite fast convergence* [8]. Furthermore, our group proposed closed form solutions for second-order digital filters [9].

So far, it has not been clarified whether the minimum L_2-sensitivity realization *subject to* L_2-scaling constraints satisfy the sufficient conditions for the absence of limit cycles. We can show that sufficient conditions (13) in Theorem 1 and (16) in Theorem 4 do not hold for the minimum L_2-sensitivity realizations *subject to* L_2-scaling constraints in general. Therefore, we have two other ways to prove the absence of limit cycles: examining sufficient conditions (14) in Theorem 2 or (15) in Theorem 3.

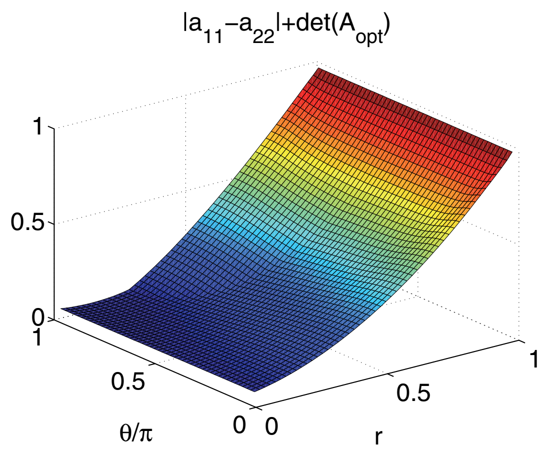


Fig. 3. Values of $|a_{11} - a_{22}| + \det(\boldsymbol{A}_{\text{opt}})$ of the minimum L_2-sensitivity realizations subject to L_2-scaling constraints of second-order digital filters for various poles (λ_1, λ_2) in Eq. (23).

IV. Numerical Examples

For second-order digital filters, we can synthesize the minimum L_2-sensitivity realizations *subject to* L_2-scaling constraints $(\boldsymbol{A}_{\text{opt}}, \boldsymbol{b}_{\text{opt}}, \boldsymbol{c}_{\text{opt}}, d_{\text{opt}})$ in closed form [9]. We can prove that the coefficient matrix $\boldsymbol{A}_{\text{opt}}$ always satisfy $a_{12}a_{21} < 0$. Therefore, it is worth examining the sufficient condition in Eq. (15b) for second-order digital filters.

Consider second-order digital filters with poles (λ_1, λ_2) and zeros (p_1, p_2) given by

$$H(z) = c\frac{(1 - p_1z^{-1})(1 - p_2z^{-1})}{(1 - \lambda_1z^{-1})(1 - \lambda_2z^{-1})} \tag{22}$$

where c is an arbitrary real scalar, which does not affect the limit cycles. We take various poles (λ_1, λ_2) and fixed zeros (p_1, p_2) as follows:

$$(\lambda_1, \lambda_2) = (re^{j\theta}, re^{-j\theta})$$

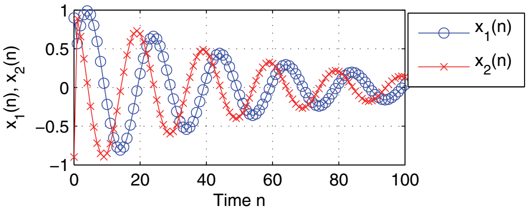
$$\text{where} \begin{cases} r = 0.01, 0.02, \cdots, 0.98, 0.99 \\ \theta = [0.02, 0.04, \cdots, 0.96, 0.98] \times \pi \end{cases} \tag{23}$$

$$(p_1, p_2) = (-1, -1) \text{ (fixed)}. \tag{24}$$

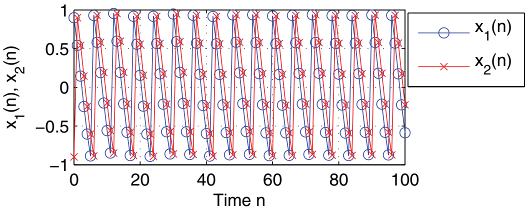
For all poles and zeros above, we synthesize the minimum L_2-sensitivity realization $(\boldsymbol{A}_{\text{opt}}, \boldsymbol{b}_{\text{opt}}, \boldsymbol{c}_{\text{opt}}, d_{\text{opt}})$ and calculate the value of $|a_{11} - a_{22}| + \det(\boldsymbol{A}_{\text{opt}})$.

The experimental result is shown in Fig. 3. The condition $|a_{11} - a_{22}| + \det(\boldsymbol{A}_{\text{opt}}) < 1$ is satisfied for all pole locations given in Eq. (23). We can see that the values of $|a_{11} - a_{22}| + \det(\boldsymbol{A}_{\text{opt}})$ mainly depend on the pole radius r, rather than the pole angle θ. As the pole radius r becomes close to 1, $|a_{11} - a_{22}| + \det(\boldsymbol{A}_{\text{opt}})$ approaches to 1. Although zeros (p_1, p_2) are fixed in this example, we have experimentally confirmed that the condition $|a_{11} - a_{22}| + \det(\boldsymbol{A}_{\text{opt}}) < 1$ is satisfied for also other set of zeros (p_1, p_2).

We next demonstrate the absence of limit cycles of the minimum L_2-sensitivity realization subject to L_2-scaling con-



(a) Minimum L_2-sensitivity realization



(b) Direct Form II

Fig. 4. Zero-input responses. $x_1(n)$ and $x_2(n)$ are state variables denoted by $\boldsymbol{x}(n) = [x_1(n)\ x_2(n)]^T$.

straints by observing zero-input responses. Consider a second-order narrow-band band-pass digital filter $H(z)$ given by

$$H(z) = \frac{0.0032 + 0.0064z^{-1} + 0.0032z^{-2}}{1 - 1.8641z^{-1} + 0.9604z^{-2}}. \tag{25}$$

The above second-order digital filter is obtained by letting poles $(\lambda_1, \lambda_2) = (0.98\exp(j0.1\pi), 0.98\exp(-j0.1\pi))$, zeros $(p_1, p_2) = (-1, -1)$, and $c = 0.0032$ in Eq. (22). The minimum L_2-sensitivity realization subject to L_2-scaling constraints is given by

$$\left[\begin{array}{c|c} \boldsymbol{A}_{\text{opt}} & \boldsymbol{b}_{\text{opt}} \\ \hline \boldsymbol{c}_{\text{opt}} & d_{\text{opt}} \end{array}\right] = \left[\begin{array}{cc|c} 0.9320 & 0.3029 & 0.0520 \\ -0.3027 & 0.9320 & 0.2765 \\ \hline 0.1416 & 0.0178 & 0.0032 \end{array}\right]. \tag{26}$$

We calculate the zero-input responses of the minimum L_2-sensitivity realization and the direct form II, with the initial state $\boldsymbol{x}(0) = [0.9, -0.9]^T$. We let the dynamic range of signals to be $[-1, 1)$ and adopt two's complement as the overflow characteristic. The zero-input responses are shown in Fig. 4(a) and 4(b). In this numerical example, the overflow of the state variables occurs in both cases. For the minimum L_2-sensitivity realization subject to L_2-scaling constraints, the state variables $x_1(n)$ and $x_2(n)$ converge to zero without limit cycles after the overflow, as shown in Fig. 4(a). On the other hand, for the Direct Form II, a large-amplitude autonomous oscillation is observed as shown in Fig. 4(a).

From these numerical results, we conjecture that the minimum L_2-sensitivity realizations *subject to* L_2-scaling constraints are free of limit cycles for second-order digital filters. However, it has not been clarified whether these results always hold or not, since they are just numerical results for specific poles. Therefore, it is our future work to give a theoretical evidence for these results.

V. Conclusions

This paper numerically showed the absence of limit cycles in second-order digital filters with minimum L_2-sensitivity *subject to* L_2-scaling constraints. We designed second-order digital filters for various pole-zero configurations, synthesized the minimum L_2-sensitivity realizations, and examined if their coefficient matrices satisfy a sufficient condition for the absence of limit cycles. Furthermore, we demonstrate the absence of limit cycles of the minimum L_2-sensitivity realizations *subject to* L_2-scaling constraints of a second-order digital filter by observing its zero-input response. From the numerical results, we conjecture that the minimum L_2-sensitivity realizations *subject to* L_2-scaling constraints of second-order digital filters are free of limit cycles in general. Our future work is giving a theoretical proof of the absence of limit cycles in the minimum L_2-sensitivity realizations *subject to* L_2-scaling constraints.

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